A lecture on

Compressible Fluid Flow

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CONTENTS

Compressible Fluid Flow: Thermodynamic relations- Continuity, Momentum and Energy equations- Velocity of sound in a compressible fluid- Mach number and its significance, Limits of incompressibility- Pressure field due to a moving source of disturbance, Propagation of pressure waves in a compressible fluids-Stagnation properties- Stagnation pressure, Temperature and density- Area velocity relationship for compressible flow.

Textbooks:

1. Fluid Mechanics and Hydraulic Machines, by R. K. Bansal, Laxmi publications.
2. Hydraulics and Fluid Mechanics - P.N. Modi, S.M. Seth 2nd edition, Standard Book House, 2005.

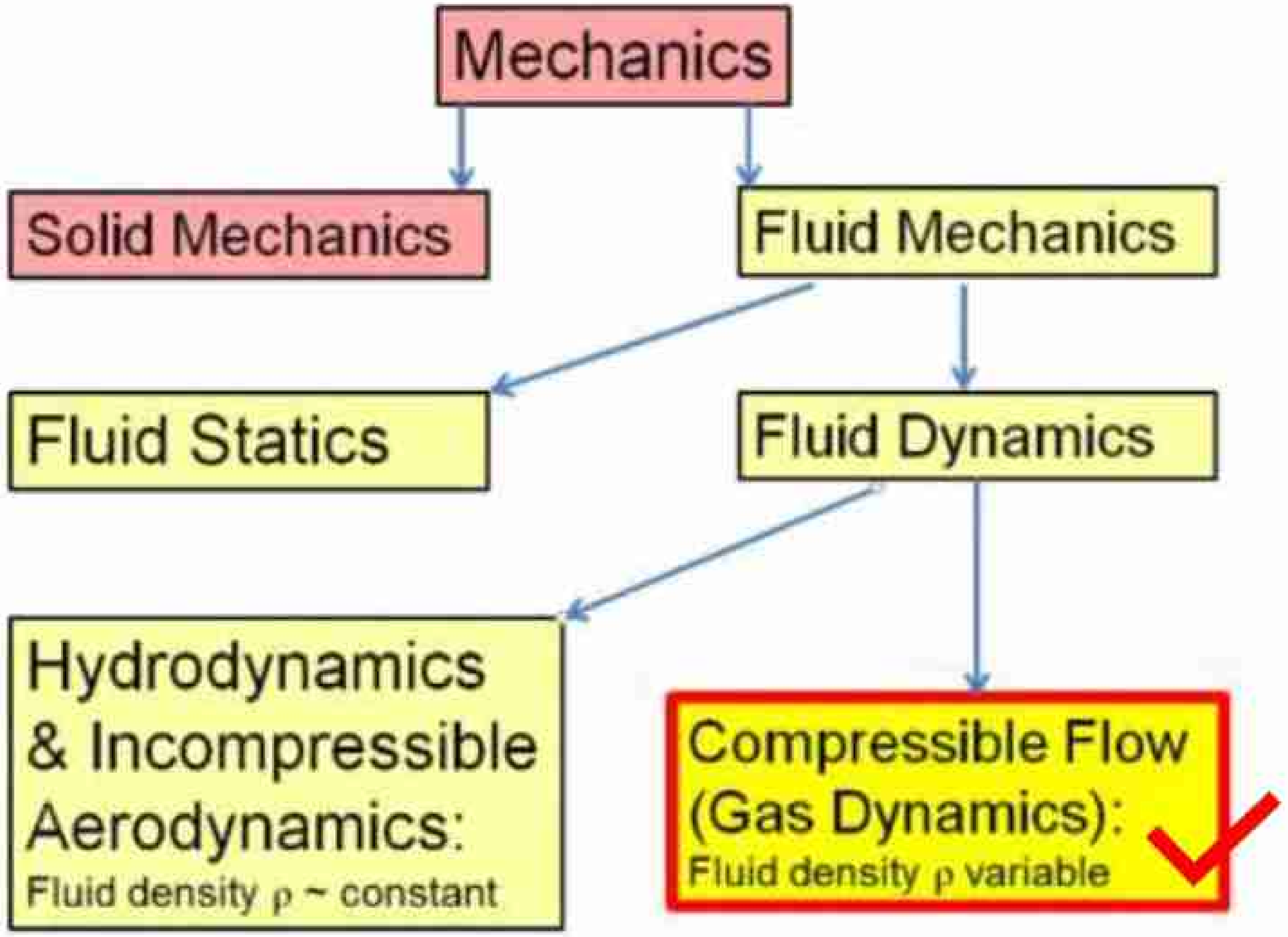
Reference Books:

1. Fluid Mechanics and Fluid Power Engineering by Dr. D.S. Kumar, S.K. Kataria&Sons.
3. Fluid Mechanics: Fundamentals and Applications (4th edition, SIE) by John. M. Cimbala Yunus A. Cengel
4. Fluid Mechanics and Hydraulic Machines by R. K. Rajput, S.Chand& Co.

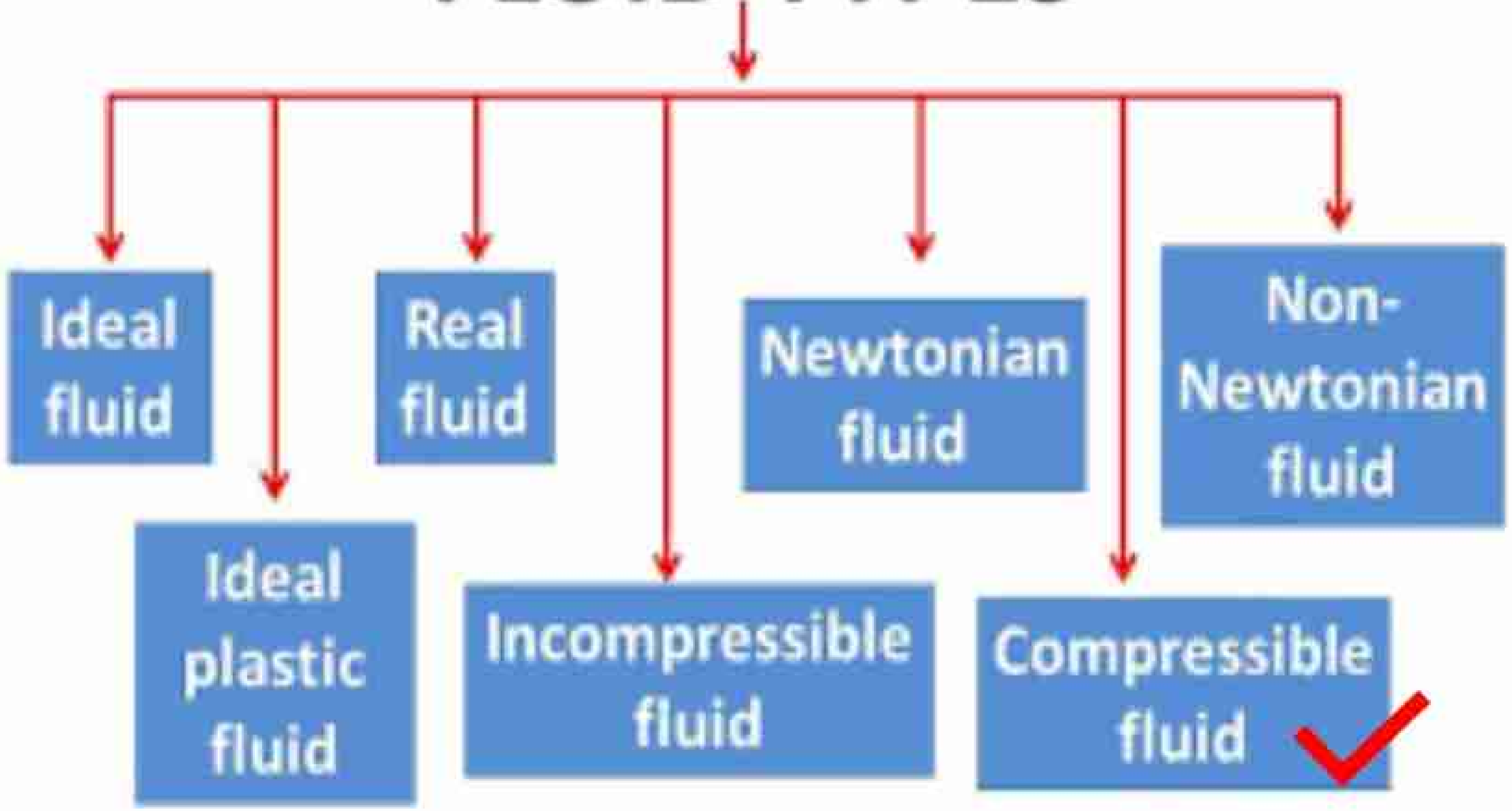
Weblinks:

1. Prof. S.K. Som, IIT Kharagpur, Fluid Mechanics & Hydraulic Machines.
   http://nptel.ac.in/courses/112105171/
2. Prof. Gowtham Biswas, IIT Kharagpur, Fluid Mechanics & Hydraulic Machines.
   http://nptel.ac.in/courses/112104118/
3. http://www.efluids.com/

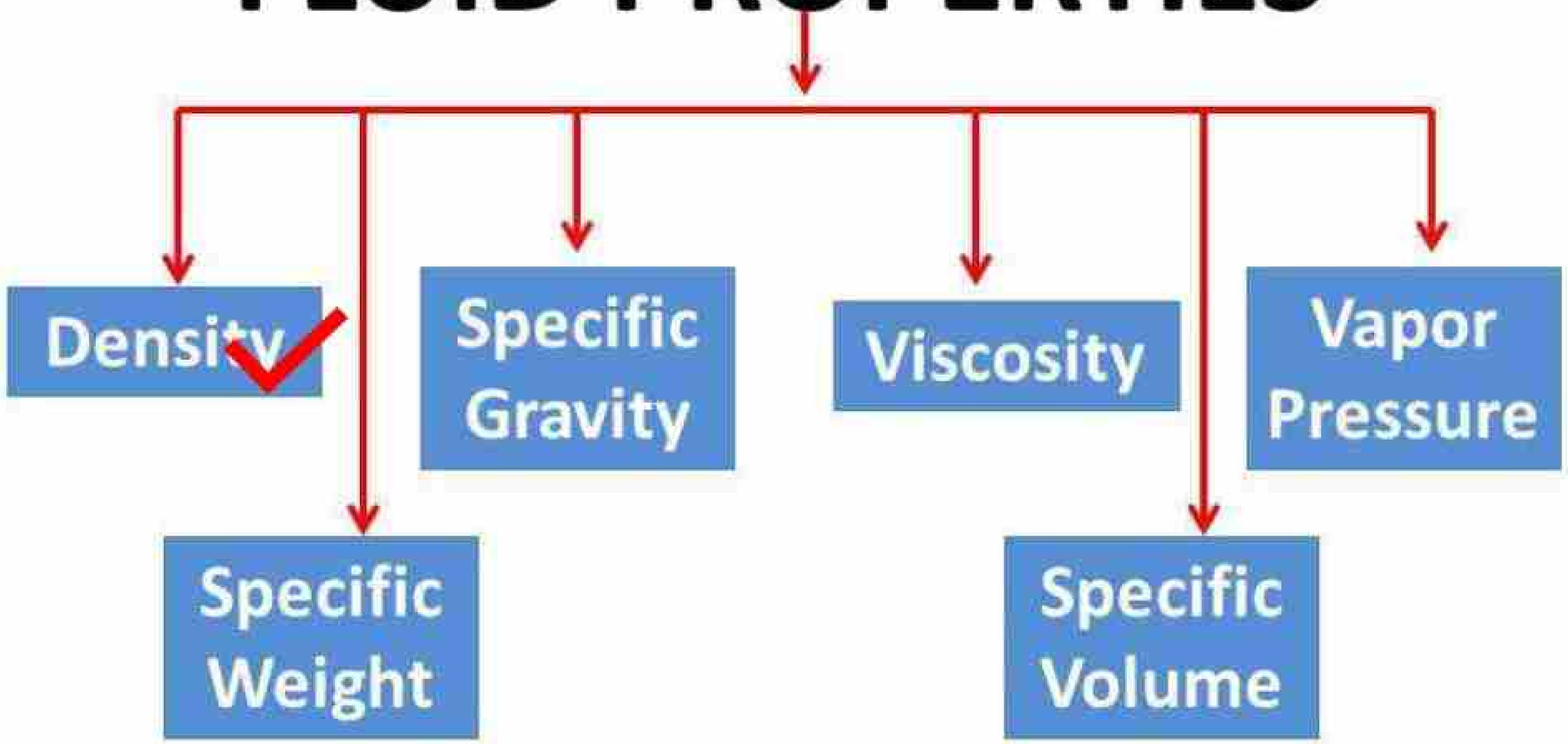

FLUID
TYPE
SOLID
LIQUID
PROPERTIES
WEIGHT DENSITY
MASS DENSITY
SPECIFIC GRAVITY
SURFACE TENSION
VAPOUR PRESSURE
COMPRESSIBILITY
VISCOSITY
FORCE
FLUID MECHANICS
HYDRO STAT ICS
STUDY OF LIQUID IN STATIC POSITION
HYDRO KINEMATICS
STUDY OF MOTION OF FLUID
HYDRO DYNAMICS
STUDY OF MOTION OF FLUID AND FORCE
IDEAL FLUID
NO VISCOSITY
NO SURFACE TENSION
NO COMPRESSIBILITY
NO SURFACE TENSION
REAL FLUID
DENSITY
VISCOSITY
SURFACE TENSION
COMPRESSIBILITY



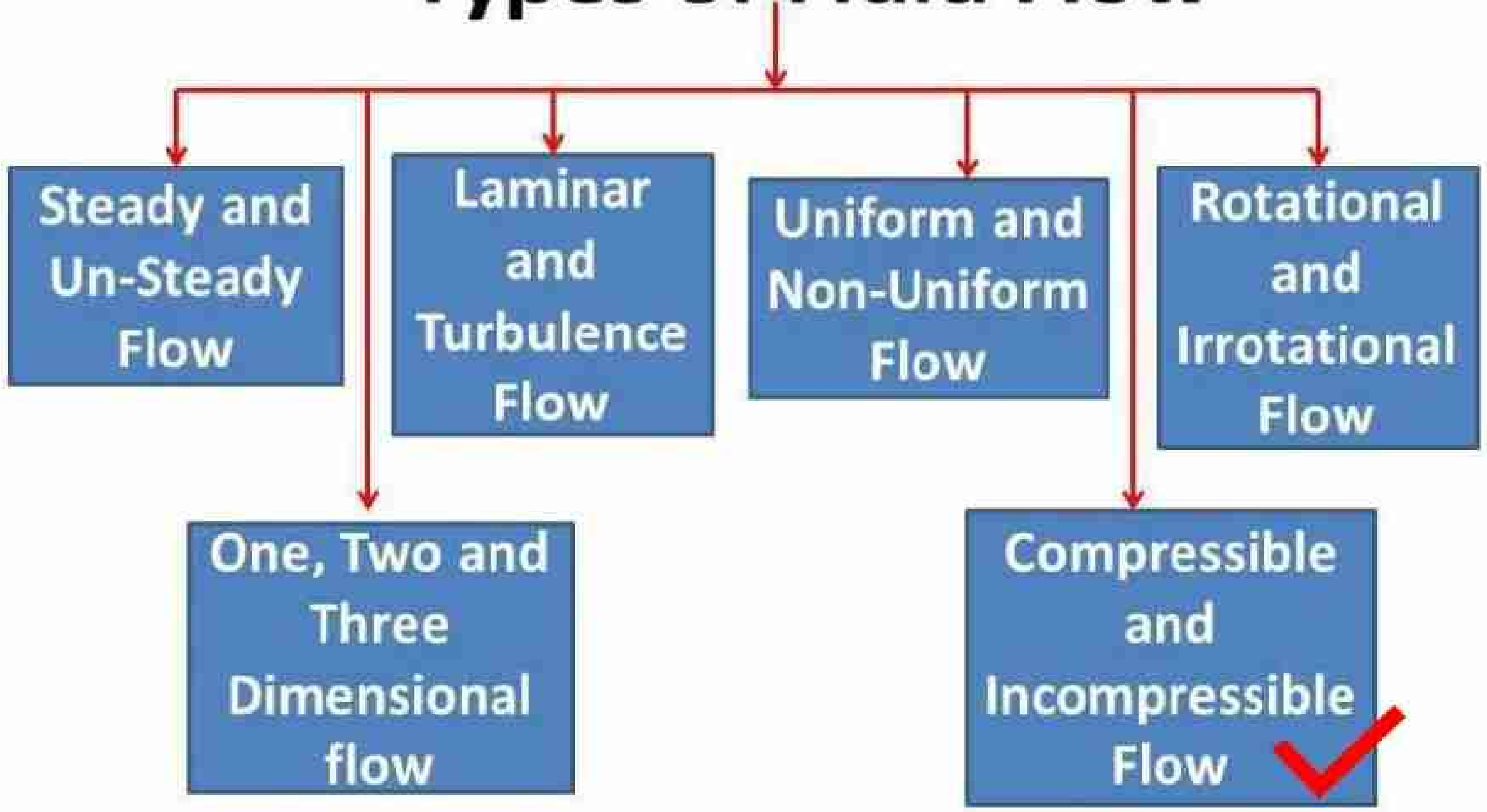
FLUID TYPES



FLUID PROPERTIES



Types of Fluid Flow



For the most part, we have limited our consideration so far to flows for which density variations and thus compressibility effects are negligible. In this chapter, we lift this limitation and consider flows that - involve significant changes in density. Such flows are called ***compressible flows***, and they are frequently encountered in devices that involve the flow of gases at very high speeds. Compressible flow combines fluid dynamics and thermodynamics in that both are necessary to the development of the required theoretical background.

All real fluids are compressible to some extent and therefore their density will change with change in pressure or temperature. If the relative change in density $\Delta \rho/\rho$ is small, the fluid can be treated as incompressible. A compressible fluid, such as air, can be considered as incompressible with constant ρ if changes in elevation are small, acceleration is small, and/ or temperature changes are negligible. In other words, if Mach's number U/C, where C is the sonic velocity, is small, compressible fluid can be treated as incompressible.

'Compressibility' affects the drag co-efficients of bodies by formation of shock waves, discharge coefficients of measuring devices such as orifice meters, venturi meters and pitot tubes, stagnation pressure and flows in converging-diverging sections. The gases are treated as compressible fluids and study of this type of flow is often referred to as '***Gas dynamics***'.

COMPRESSIBLE FLUID
VERSUS
INCOMPRESSIBLE FLUID

COMPRESSIBLE FLUID	INCOMPRESSIBLE FLUID
Compressible fluid is matter that can be compressed with the application of an external pressure	Incompressible fluid is matter that cannot be compressed with the application of an external pressure
Volume can be reduced with the application of a pressure on the fluid	Volume cannot be reduced with the application of a pressure on the fluid
Density can be changed with the application of a pressure on the fluid	Density cannot be changed with the application of a pressure on the fluid
Value of Mach number should be greater than 0.3	Value of Mach number should be less than 0.3

Some key problems where compressibility effect has to be considered are :

- Flow of gases through nozzles, orifices.
- Atmospheric sciences, ocean sciences.
- Supersonic wind tunnels
- Compressors, diffusers.
- Flight of high-speed Aeroplan's and projectiles moving at higher altitudes
- Water hammer and acoustics
- Jet engines, Rocket motors, gas turbines, steam turbines, IC engine etc.,.
- High speed entry into a planetary atmosphere.
- Gas pipelines
- Casting
- Commercial applications like abrasive blasting
- Understanding nature like motion of snails.

BASIC THERMODYNAMIC RELATIONS

1. The Characteristics Equation of State

At temperatures that are considerably in excess of critical temperature of a fluid, and at very low pressure, the vapour of fluid tends to obey the equation:

In practice, no gas obeys this law rigidly, but many gases tend towards it. An imaginary ideal gas which obeys this law is called a *perfect gas,* and the equation $\frac{pv}{T} = R$, is called the *characteristic equation of a state of a perfect gas*. The constant R is called the *gas constant*. Each perfect gas has a different gas constant.

Units of R are Nm/kg K or kJ/kg K

Usually, the characteristic equation is written as :

$$pv = RT \qquad \text{...(1)}$$

or, for m kg, occupying V m^3,

$$pV = mRT \qquad \text{...(2)}$$

or,

$$p = \frac{m}{V} RT = \rho RT \qquad \text{...(2 (a))}$$

Taking log on both sides, we get:

$$\ln(p) = \ln(\rho) + \ln(R) + \ln(T)$$

Upon differentiation, we have:

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dR}{R} + \frac{dT}{T}$$

Since R is constant for a particular gas, its derivative is zero.

$$\therefore \quad \frac{dp}{p} - \frac{d\rho}{\rho} - \frac{dT}{T} = 0 \qquad ...(3)$$

Eqn. (3) is the *differential equation of a perfect gas.*

2. Specific Heats

— The specific heat of a solid or liquid is usually defined as *the heat required to raise unit mass through one degree temperature rise.*

— For a gas there are an infinite number of ways in which heat may be added between any two temperatures, and hence a gas could have an infinite number of specific heats. However, only two specific heats for gases are defined.

(*i*) Specific heat at constant volume, c_v

(*ii*) Specific head at constant pressure, c_p.

(In case of real gases, c_p and c_v *vary with temperature,* but a suitable *average value* may be used for most practical purposes.)

$$c_p = c_v + R \quad \text{...(4)}$$

$$\frac{c_p}{c_v} = \gamma \text{ (gamma)} \quad \text{...(5)}$$

3. Internal Energy

It is the heat energy stored in a gas. If a certain amount of heat is supplied to a gas the result is that temperature of gas may increase or volume of gas may increase thereby doing some external work or both temperature and volume may increase. *If during heating of a gas the temperature increases its internal energy will also increase.*

Joule's law *of internal energy states that the internal energy of a perfect gas is a function of temperature only.* In other words, internal energy of a gas is dependent on the *temperature change only* and is *not affected by the change in pressure and volume.*

We do not know how to find the absolute quantity of internal energy in any substance, however, what is needed in engineering is the change of internal energy (ΔU).

4. Enthalpy

One of the fundamental quantities which occurs invariably in thermodynamics is the *sum of internal energy (u) and pressure volume product (pv). This sum is called* **Enthalpy (h)**.

i.e. $$h = u + pu$$

The total enthalpy of mass, *m, of a fluid is given by,*

$$\boxed{H = U + pV}, \text{ where } H = mh$$

Energy. Energy is a general term embracing *energy in transition and stored energy*. The stored energy of a substance may be in the forms of *mechanical energy* and *internal energy* (other forms of stored energy may be chemical energy and electrical energy). Part of the stored energy may take the form of either potential energy or kinetic energy due to velocity. The balance part of the energy is known as *internal energy*.

Heat and work. These are the forms of energy in transition and are the only forms in which energy can cross the boundaries of a system. Neither heat nor work can exist as stored energy.

Work. Work is said to be done when a *force moves through a distance.* If a part of the boundary of a system undergoes a displacement under the action of a pressure, the work done W is the product of the force (pressure × area) and the distance it moves in the direction of the force.

Work is a transient quantity which only appears at the boundary while a change of state is taking place within a system. Work is 'something' which appears at the boundary when a system changes its state due to the movement of a part of the boundary under the action of a force.

Work output of the system $= +W$ ✓

Work input to system $= -W$ ✓

Heat. Heat (denoted by the symbol Q) may be defined in an analogous way to work as follows:

"*Heat is something which appears at the boundary when a system changes its state due to a difference in temperature between the system and its surroundings*".

Heat, like work, is a transient quantity which only appears at the boundary while a change is taking place within the system.

Heat received by the system $= +Q$ ✓

Heat rejected or given up by the system $= -Q$ ✓

Isothermal process pv or $\frac{p}{\rho}$ = constant, T = constant). A process at a constant temperature is called an *isothermal process. When a working substance in a cylinder behind a piston expands from a high pressure there is a tendency for the temperature to fall. In an isothermal expansion heat must be added continuously in order to keep the temperature at the initial value. Similarly in an isothermal compression heat must be removed from the working substance continuously during the process.*

Formulae (for unit mass) :

Heat added, $$Q = p_1 v_1 \ln\left(\frac{v_2}{v_1}\right) = RT_1 \ln\left(\frac{\rho_1}{\rho_2}\right) \quad \text{...(13)}$$

Work done, $$W = p_1 v_1 \ln\left(\frac{v_2}{v_1}\right) = RT_1 \ln\left(\frac{\rho_1}{\rho_2}\right) \quad \text{...(14)}$$

p, v, T, relations : $$p_1 v_1 = \left(\text{or } \frac{p_1}{\rho_1} = \frac{p_2}{\rho_2}\right) \quad \text{...(15)}$$

Adiabatic process (pv^{γ} or $\frac{p}{\rho^{\gamma}}$ = constant). An adiabatic process is one *in which no heat is transferred to or from the gas* during the process. Such a process can be reversible or irreversible. *For an adiabatic process to take place, perfect thermal insulation for the system must be available.*

Formulae (for unit mass) :

Heat added, $Q = 0$ ...(16)

Work done, $W = \frac{p_1v_1 - p_2v_2}{\gamma - 1} = \frac{R(T_1 - T_2)}{\gamma - 1}$ ...(17)

p, v, T, relations : $p_1v_1^{\gamma} = p_2v_2^{\gamma}$ ...(18)

$$\frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{\gamma - 1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma - 1}{\gamma}} \quad ...(19)$$

If the adiabatic process is *reversible* (or frictionless), it is known as *isentropic process*. In case the pressure and density are related in such a way that $\gamma \neq \frac{c_p}{c_v}$ but is equal to some positive value then the process is known as *polytropic*, according to which $\frac{p}{\rho^n}$ = constant ($n \neq \gamma$).

BASIC EQUATIONS OF COMPRESSIBLE FLUID FLOW

The basic equations of compressible fluid flow are :

(i) Continuity equation ✓

(ii) Momentum equation ✓

(iii) Energy equation ✓

(iv) Equation of state ✓

The only change from incompressible fluid cases is that thermodynamic laws are applied in addition to the basic principle of conservation of mass, energy and momentum.

1. Continuity Equation

In case of *one-dimensional flow,* mass per second = ρAV

(where, ρ = mass density, A = area of cross-section, V = velocity)

Since the mass or mass per second is constant according to law of conservation of mass, therefore,

$$\rho AV = \text{Constant} \qquad \text{...(20)}$$

Differentiating the above equation, we get:

$$d(\rho AV) = 0 \quad \text{or} \quad \rho d(AV) + AV d\rho = 0$$

or, $$\rho(AdV + VdA) + AVd\rho = 0 \quad \text{or} \quad \rho AdV + \rho VdA + AVd\rho = 0$$

Dividing both sides by ρAV, and rearranging we get:

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \qquad \text{...(21)}$$

Eqn. (15.18) is also known as equation of continuity in *differential form.*

2. Momentum Equation

The momentum equation for compressible fluids is similar to the one for incompressible fluids. This is because in momentum equation the *change in momentum flux is equated to force required to cause this change.*

Momentum flux = Mass flux × velocity $= \rho AV \times V$

But the mass flux *i.e.* ρAV = constant ...By continuity equation

Thus the momentum equation is completely independent of the compressibility effects and hence for compressible fluids too the momentum equation, say X-direction, may be expressed as :

$$\Sigma F_x = (\rho AVV_x)_2 - (\rho AVV_x)_1 \quad ...(22)$$

3. Bernoulli's or Energy Equation

In chapter 6 Bernoulli's equation for an incompressible fluid has been derived and the same procedure is followed. As the flow of compressible fluid is steady, the same Euler equation (Eqn. 6.) is obtained as :

$$\frac{dp}{\rho} + VdV + gdz = 0 \quad ...(.23)$$

Integrating both sides, we get:

$$\int \frac{dp}{\rho} + \int V dV + \int g dz = \text{constant}$$

or,

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \qquad \text{...(24)}$$

In compressible flow since ρ is not constant it cannot be taken outside the integration sign. In compressible fluids the pressure (p) changes with change of density (ρ), depending on the type of process. Let us find out the Bernoulli's equation for isothermal and adiabatic processes.

(*a*) Bernoulli's equation for isothermal process :

In case of an *isothermal process*

$$pv = \text{constant} \quad \text{or} \quad \frac{p}{\rho} = \text{constant} = c_1 \ (say)$$

(where v = specific volume = $1/\rho$)

$$\therefore \qquad \rho = \frac{p}{c}$$

Hence, $\int \frac{dp}{\rho} = \int \frac{dp}{p/c_1} = \int \frac{c_1 dp}{p} = c_1 \int \frac{dp}{p} = c_1 \ln(p) = \frac{p}{\rho} \ln(p) \quad \left(\because \; c_1 = \frac{p}{\rho} \right)$

Substituting the value of $\int \frac{dp}{\rho}$ in eqn. (24), we get

$$\frac{p}{\rho} \ln(p) + \frac{V^2}{2} + gz = \text{constant}$$

Dividing both sides by g, we get

$$\frac{p}{\rho g} \ln(p) + \frac{V^2}{2g} + z = \text{constant} \qquad ...(25)$$

Eqn. (25) is the *Bernoulli's equation for compressible flow undergoing isothermal process*.

(*b*) Bernoulli's equation for adiabatic process :

In case of an *adiabatic process*:

$$pv^{\gamma} = \text{constant} \quad \text{or} \quad \frac{p}{\rho^{\gamma}} = \text{constant} = c_2 \, (say)$$

$$\therefore \qquad \rho^{\gamma} = \frac{p}{c_2} \qquad \text{or} \qquad \rho = \left(\frac{p}{c_2}\right)^{1/\gamma}$$

Hence,

$$\int \frac{dp}{\rho} = \int \frac{dp}{(p / c_2)^{1/\gamma}} = (c_2)^{1/\gamma} \int \frac{1}{p^{1/\gamma}} \, dp = (c_2)^{1/\gamma} \int p^{-1/\gamma} \, dp$$

$$= (c_2)^{1/\gamma} \left[\frac{p^{-\frac{1}{\gamma}+1}}{\left(-\frac{1}{\gamma}+1\right)} \right] = \frac{(c_2)^{1/\gamma} \, (p)^{\left(\frac{\gamma-1}{\gamma}\right)}}{\left(\frac{\gamma-1}{\gamma}\right)} = \frac{\gamma}{\gamma-1} \, c_2^{1/\gamma} \, (p)^{\left(\frac{\gamma-1}{\gamma}\right)}$$

$$= \left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p}{\rho^{\gamma}}\right)^{1/\gamma} (p)^{\left(\frac{\gamma-1}{\gamma}\right)} \qquad \left(\because c_2 = \frac{p}{\rho^{\gamma}}\right)$$

$$= \left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p^{1/\gamma}}{\rho^{\gamma\times\frac{1}{\gamma}}}\right)(p)^{\left(\frac{\gamma-1}{\gamma}\right)}$$

$$= \left(\frac{\gamma}{\gamma-1}\right)\frac{p^{\left(\frac{1}{\gamma}+\frac{\gamma-1}{\gamma}\right)}}{\rho} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho}$$

Substituting the value of $\int \frac{dp}{\rho}$ in eqn. (15.24), we get

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

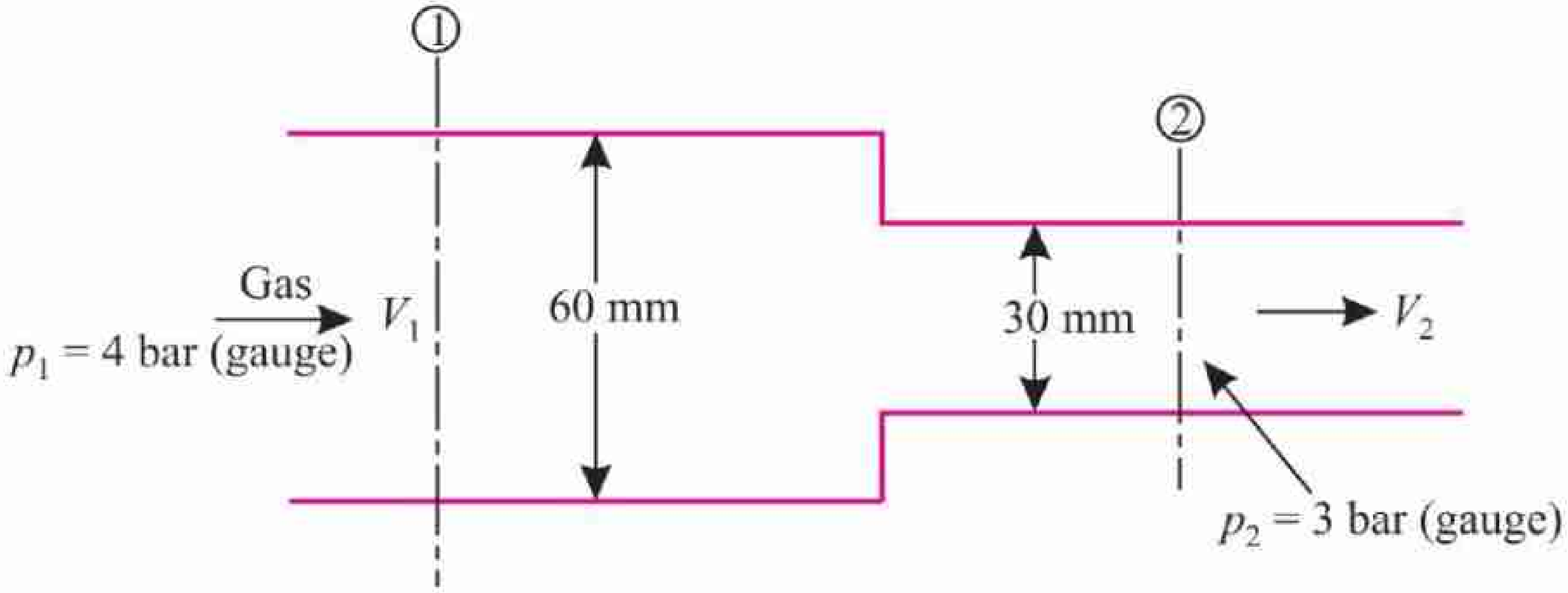
Dividing both sides by g, we get

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant} \qquad ...(26)$$

Eqn. (26) is the *Bernoulli's equation for compressible flow undergoing adiabatic process*.

Example 1 *Fig. shows a horizontal pipe in which gas is flowing at a temperature of 6ºC. The pressures at the sections 1 and 2 are 4 bar (gauge) and 3 bar (gauge) respectively. If R = 287 J/kg K and atmospheric pressure is 1 bar find the velocities of the gas at these sections.*

Solution.



Section 1: Diameter of pipe, $D_1 = 60 \text{ mm} = 0.06 \text{ m}$

$$\therefore \quad \text{Area, } A_1 = \frac{\pi}{4} \times 0.06^2 = 2.827 \times 10^{-3} \text{ m}^2$$

$$\text{Temperature, } T_1 = 6 + 273 = 279 \text{ K}$$

$$\text{Pressure, } p_1 = 4 \text{ bar (gauge)} = 4 + 1 = 5 \text{ bar (abs.)} = 5 \times 10^5 \text{ N/m}^2 \text{ (abs.)}$$

Section 2: Diameter of pipe, $D_2 = 30 \text{ mm} = 0.03 \text{ m}$

$$\therefore \quad \text{Area, } A_2 = \frac{\pi}{4} \times 0.3^2 = 7.0686 \times 10^{-4} \text{ m}^2$$

$$\text{Pressure, } p_2 = 3 \text{ bar (gauge)}$$

$$= 3 + 1 = 4 \text{ bar (abs.)} = 4 \times 10^5 \text{ N/m}^2 \text{ (abs.)}$$

$$\text{Gas constant, } R = 287 \text{ J/kg K}$$

Velocities of the gas at sections 1 and 2, V_1, V_2:

Applying continuity equation at 1 and 2, we get:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad \text{or} \quad \frac{V_2}{V_1} = \frac{\rho_1 A_1}{\rho_2 A_2} = \frac{\rho_1 \times 2{\cdot}827 \times 10^{-3}}{\rho_2 \times 7{\cdot}0686 \times 10^{-4}} = 4 \times \frac{\rho_1}{\rho_2} \qquad ...(i)$$

For *an isothermal process,* we have:

$$\frac{p_1}{\rho_1} = \frac{p_2}{\rho_2} \quad \text{or} \quad \frac{\rho_1}{\rho_2} = \frac{p_1}{p_2} = \frac{5 \times 10^5}{4 \times 10^5} = 1.25$$

Substituting the value of $\frac{\rho_1}{\rho_2} = 1.25$ in eqn. (*i*), we get:

$$\frac{V_2}{V_1} = 4 \times 1.25 = 5 \quad \text{or} \quad V_2 = 5V_1 \qquad ...(ii)$$

Applying Bernoulli's equation at sections 1 and 2 for isothermal process (Eqn .25), we have:

$$\frac{p_1}{\rho_1 g} \ln(p_1) + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho_2} \ln(p_2) + \frac{V_2^2}{2g} + z_2$$

But, $z_1 = z_2$, since the pipe is horizontal,

$$\therefore \quad \frac{p_1}{\rho_1 g}\ln(p_1) + \frac{V_1^2}{2g} = \frac{p_2}{\rho_2}\ln(p_2) + \frac{V_2^2}{2g}$$

or, $$\frac{p_1}{\rho_1 g}\ln(p_1) - \frac{p_2}{\rho_2 g}\ln(p_2) = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

But, $\frac{p_1}{\rho_1} = \frac{p_2}{\rho_2}$ (for an *isothermal process*),

$$\therefore \quad \frac{p_1}{\rho_1 g}\ln(p_1) - \frac{p_1}{\rho_1 g}\ln(p_2) = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

or, $$\frac{p_1}{\rho_1 g}\ln\left(\frac{p_1}{p_2}\right) = \frac{(5V_1)^2}{2g} - \frac{V_1^2}{2g} = \frac{24V_1^2}{2g} = \frac{12V_1^2}{g} \qquad (\because V_2 = 5V_1)$$

or, $$\frac{p_1}{\rho_1 g}\ln\left(\frac{5\times 10^5}{4\times 10^5}\right) = \frac{12V_1^2}{g} \quad \text{or} \quad 0.223\,\frac{p_1}{\rho_1 g} = \frac{12V_1^2}{g}$$

or, $$\frac{p_1}{\rho_1} = \frac{12V_1^2}{0.223} = 53.8V_1^2 \qquad \text{...(iii)}$$

From equation of state, we have:

$$p_1 = \rho_1 R T_1 \quad \text{...Section 1} \quad \text{or} \quad \frac{p_1}{\rho_1} = RT_1 = 287 \times 279 = 80073$$

Substituting the valve of $\frac{p_1}{\rho_1}$ in eqn. (*iii*), we get:

$$53.8\, V_1^2 = 80073 \quad \text{or} \quad V_1^2 = \frac{80073}{53.8} = 1488.34$$

or, $\boldsymbol{V_1 = 38.58}$ **m/s** **(Ans.)**

From eqn. (*ii*) we have: $V_2 = 5\, V_1 = 5 \times 38.58 = \mathbf{192.9}$ **m/s** **(Ans.)**

Example 2. *In the case of air flow in a conduit transition, the pressure, velocity and temperature at the upstream section are 35 kN/m², 30 m/s and 150°C respectively. If at the downstream section the velocity is 150 m/s, determine the pressure and the temperature if the process followed is isentropiic. Take γ = 1.4, R = 290 J/kg K.*

Solution.

Section 1 (upstream) : Pressure, $p_1 = 35\ \text{kN/m}^2$,

Velocity, $V_1 = 30$ m/s

Temperature, $T_1 = 150 + 273 = 423$ K

Velocity, $V_2 = 150$ m/s

$R = 290$ J/kg K, $\gamma = 1.4$

Section 2 (downstream) :

Pressure, p_2 :

Applying Bernoulli's equation at sections 1 and 2 for *isentropic* (*reversible adiabatic*) *process*, we have:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1 g}+\frac{V_1^2}{2g}+z_1 = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2$$

Assuming $z_1 = z_2$, we have:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1 g}+\frac{V_1^2}{2g} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2 g}+\frac{V_2^2}{2g}$$

Cancelling 'g' on both the sides, and rearranging we get:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left(1-\frac{p_2}{p_1}\times\frac{\rho_1}{\rho_2}\right) = \frac{V_2^2}{2}-\frac{V_1^2}{2} \qquad \ldots(i)$$

For an isentropic flow: $\frac{p_1}{\rho_1^\gamma} = \frac{p_2}{\rho_2^\gamma}$ or $\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^\gamma$ or $\frac{\rho_1}{\rho_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}$

Substituting the value of $\frac{\rho_1}{\rho_2}$ in eqn. (i), we have:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left\{1-\frac{p_2}{p_1}\times\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}\right\} = \frac{V_2^2}{2}-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left\{1-\left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right\} = \frac{V_2^2}{2}-\frac{V_1^2}{2}$$

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1}\left\{1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right\} = \frac{V_2^2}{2}-\frac{V_1^2}{2}$$

Substituting the values, we get:

$$\frac{1.4}{1.4-1}\times 122670\left\{1-\left(\frac{p_2}{p_1}\right)^{\frac{1.4-1}{1.4}}\right\}=\frac{150^2}{2}-\frac{30^2}{2}=10800$$

$$\left(\because \frac{p_1}{\rho_1} = RT_1 = 290\times 423 = 122670\right)$$

$$429345\left\{1-\left(\frac{p_2}{p_1}\right)^{0.2857}\right\} = 10800$$

or, $$\left(\frac{p_2}{p_1}\right)^{0.2857} = 1-\frac{10800}{429345} = 0.9748$$

or, $$\frac{p_2}{p_1} = (0.9748)^{1/02857} = (0.9748)^{3.5} = 0.9145$$

or, $$p_2 = 35\times 0.9145 = \mathbf{32\ kN/m^2}$$ **(Ans.)**

Temperature, T_2:

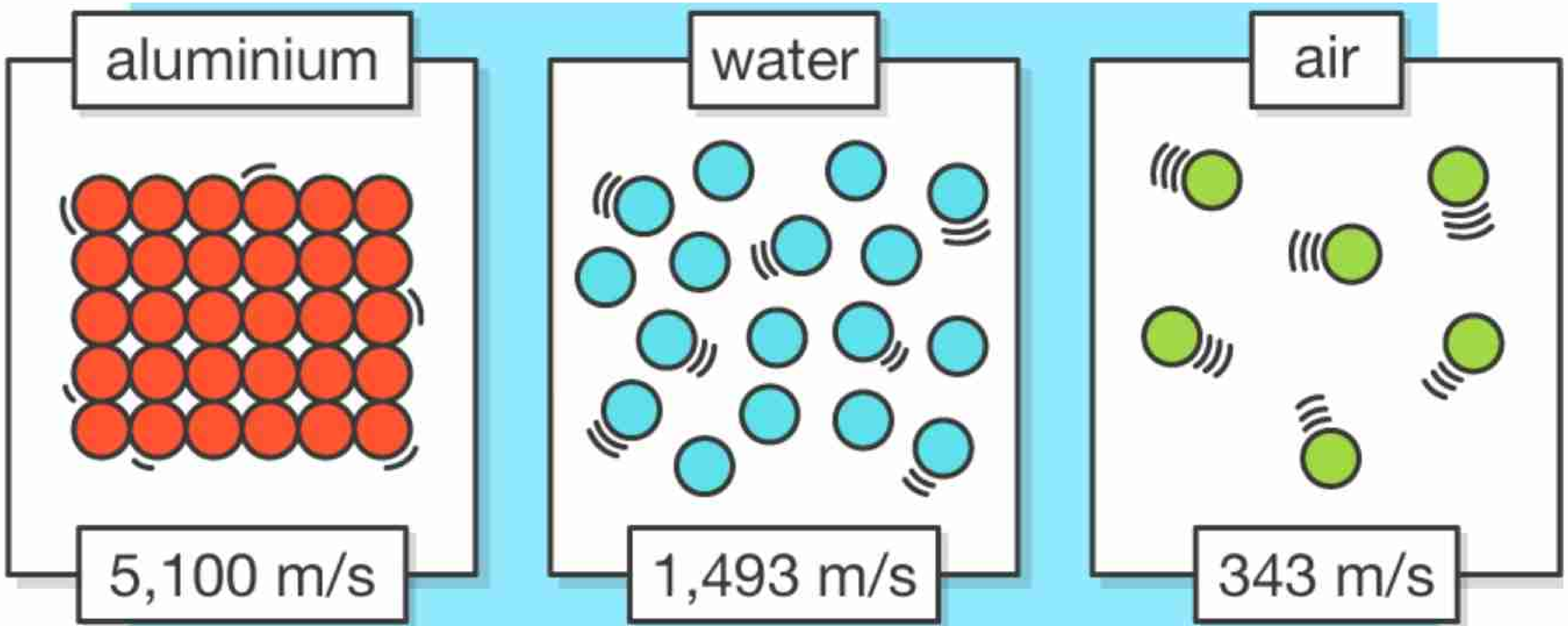
For an *isentropic process*, we have:

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma - 1}{\gamma}} = (0.9145)^{\frac{1.4-1}{1.4}} = (0.9145)^{0.2857} = 0.9748$$

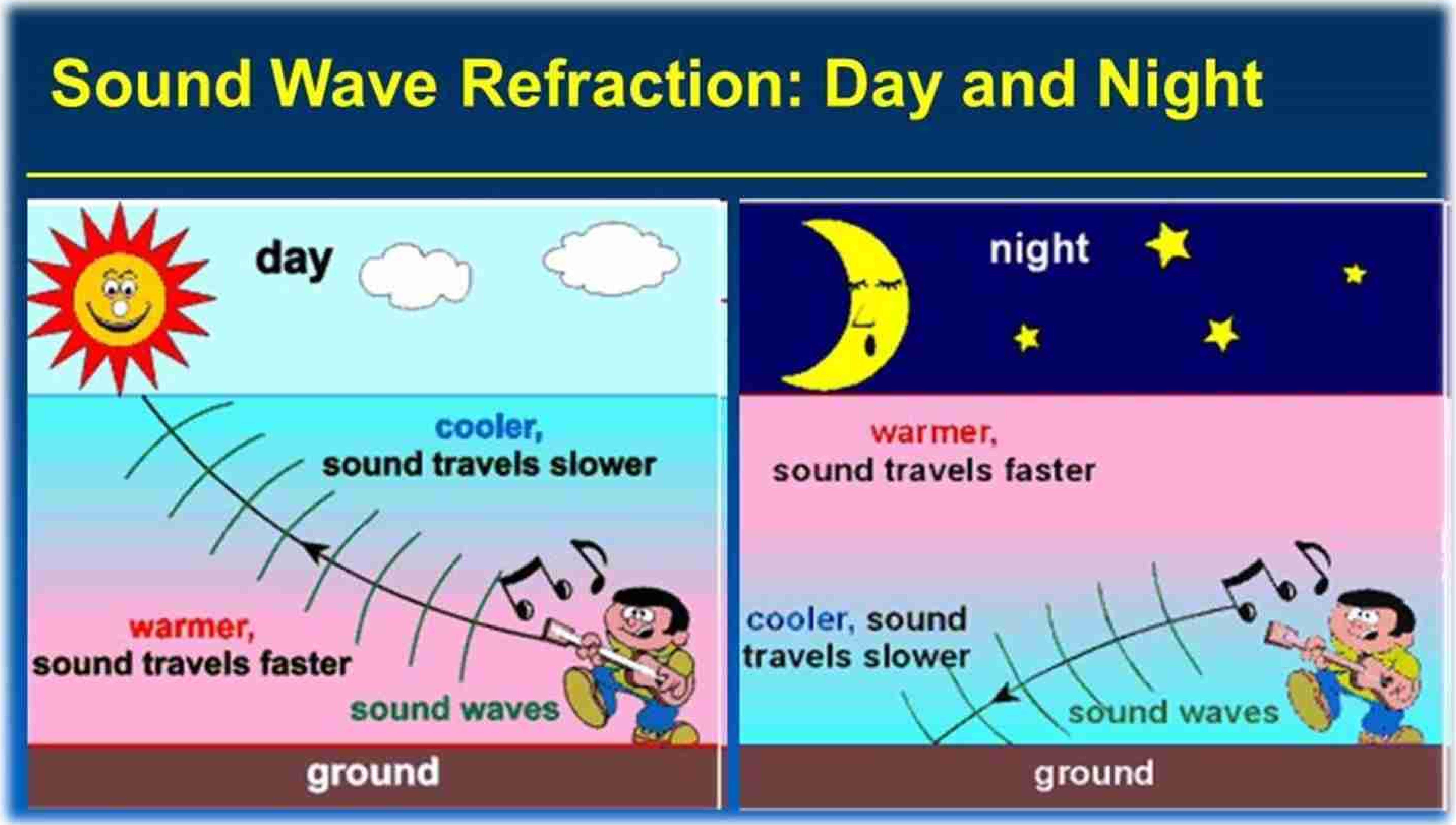
$$\therefore \quad T_2 = 423 \times 0.9748 = 412.3 \text{ K or } t_2 = 412.3 - 273 = \mathbf{139.3° C} \text{ (Ans.)}$$

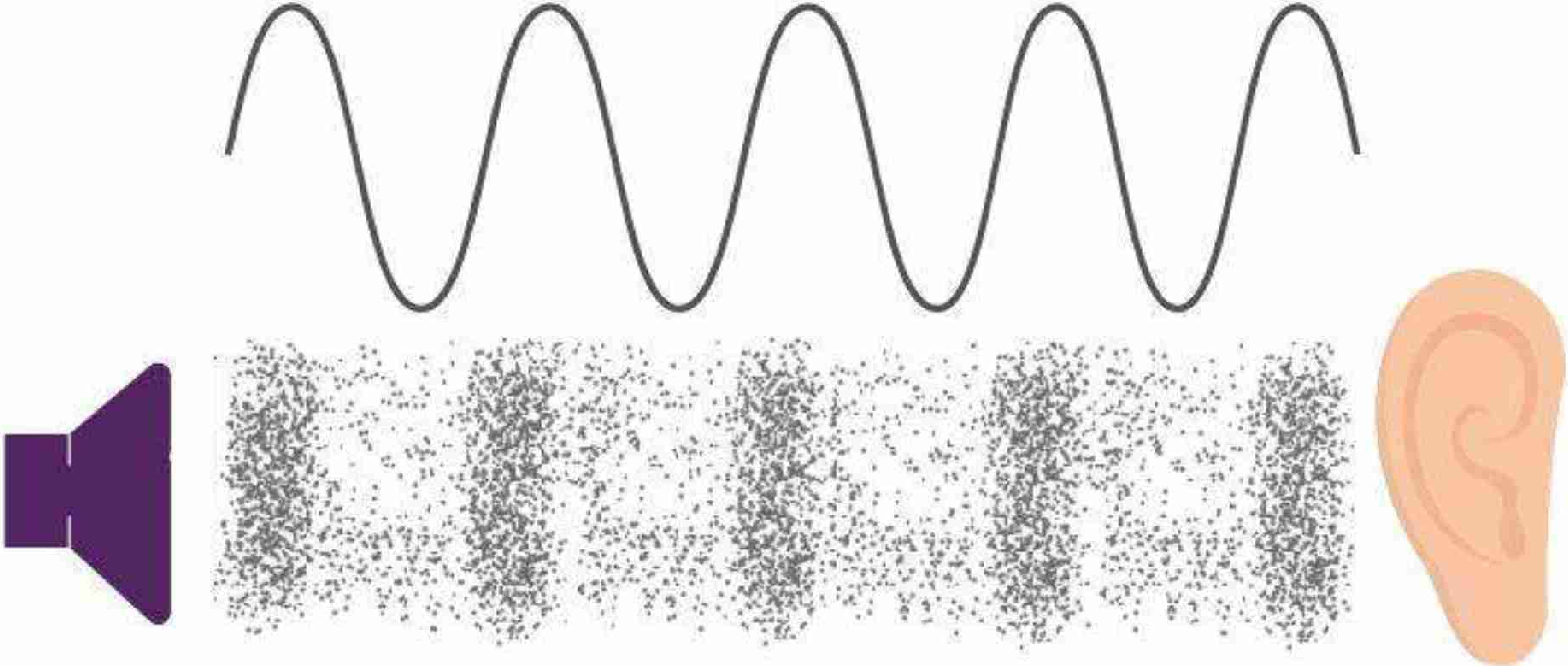
PROPAGATION OF DISTURBACES IN FLUID AND VELOCITY OF SOUND

The solids as well as fluids consist of molecules. Whereas the molecules in solids are close together , these are relatively apart in fluids. Consequently, whenever there is a minor disturbance, it travels instantaneously in case of solids; but in case of fluid the molecules change in position before the transmission or propagation of the disturbance depends upon its ***elastic properties***. ***The velocity of disturbance depends upon the changes of pressure and density of the fluid.***



The propagation of disturbance is similar to the propagation of sound through a media. *The speed of propagation of sound in a media is known as* ***acoustic or sonic velocity*** and depends upon the difference of pressure. Incompressible flow, velocity of sound (sonic velocity) is of paramount importance





Propagation of sound through fluids. The places with high density of balls is experiencing Compression and the empty spaces in between are undergoing Rarefaction. The wavelength is the graph of the pressure variation.

Derivation of Sonic Velocity (velocity of sound)

Consider a one dimensional flow through long straight rigid pipe of uniform cross-sectional area fitted with a frictionless piston at one end as shown in Fig. The tube is filled with a compressible fluid initially at rest. If the piston is moved suddenly to the right with velocity, a pressure wave would be propagated through the fluid with a velocity of sound wave.

Let,

A = Cross-sectional area of the pipe,

V = Piston velocity,

p = Fluid pressure in the pipe before the piston movement,

ρ = Fluid density before the piston movement,

dt = A small interval of time during which piston moves,

C = Velocity of pressure wave or sound wave (travelling in the fluid).

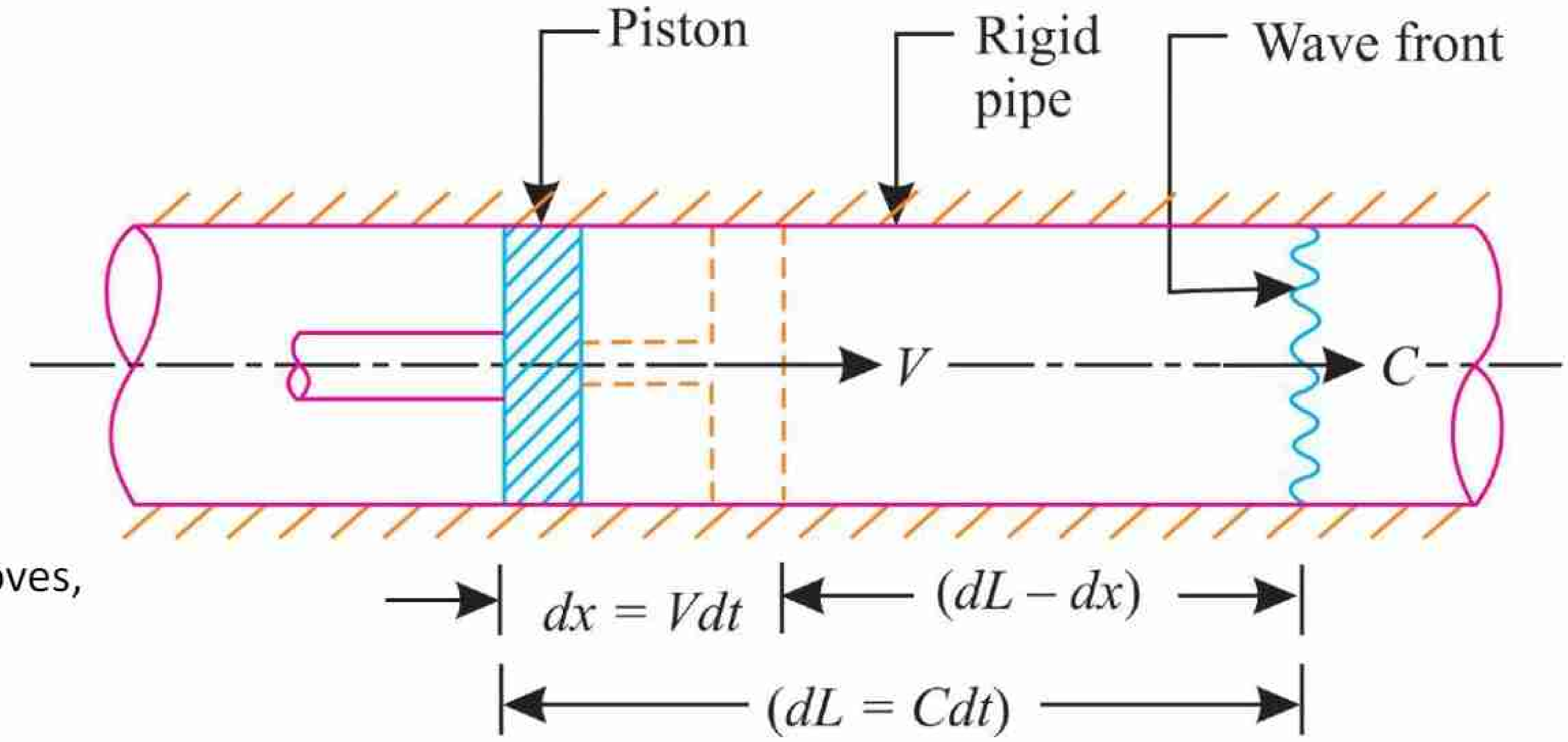


Fig. One dimensional pressure wave propagation.

Before the movement of the poston the length dL has an initial density ρ, and its total mass

$$= \rho \times dL \times A$$

When the piston moves through a distance dx, the fluid density within the compressed region of length $(dL - dx)$ will be increased and becomes $(\rho + d\rho)$ and subsequently the total mass of fluid in the compressed region $= (\rho + d\rho)(dL - dx) \times A$

$$\therefore \quad \rho \times dL \times \cancel{A} = (\rho + d\rho)(dL - dx) \times \cancel{A}$$ ...by *principle of continuity*.

But, $dL = C\,dt$ and $dx = V dt$; therefore, the above equation becomes:

$$\rho C dt = (\rho + d\rho)(C - V)\,dt$$

or, $\rho C = (\rho + d\rho)(C - V)$ or $\rho C = \rho C - \rho V + d\rho.C - d\rho.V$

or, $0 = -\rho V + d\rho.C - d\rho.V$

Neglecting the term $d\rho.V$ (V being much smaller than C), we get:

$$d\rho.C = \rho V \quad \text{or} \quad \boxed{C = \frac{\rho V}{d\rho}} \qquad ...(.27)$$

Further in the region of compressed fluid, the fluid particles have attained a velocity which is apparently equal to V (velocity of the piston), accompanied by an increase in pressure dp due to sudden motion of the piston. Applying inpulse-momentum equation for the fluid in the compressed region during dt, we get:

$$dp \times \cancel{A} \times dt = \rho \times dL \times \cancel{A} (V - 0)$$

(Force on the fluid) (Rate of change of momentum)

or, $$dp = \rho \frac{dL}{dt} V = \rho \times \frac{Cdt}{dt} \times V = \rho C V \qquad (\because dL = Cdt)$$

or, $$\boxed{C = \frac{dp}{\rho V}} \qquad ...(28)$$

Multiplying eqns. (27) and (28), we get:

$$C^2 = \frac{\rho V}{d\rho} \times \frac{dp}{\rho V} = \frac{dp}{d\rho}$$

$$\therefore \quad \boxed{C = \sqrt{\frac{dp}{d\rho}}} \qquad ...(.29)$$

Sonic Velocity in terms of Bulk Modulus

The bulk modulus of elasticity of fluid (K) is defined as:

$$K = \frac{dp}{\left(-\frac{dv}{v}\right)} \quad \text{...}(i)$$

where, dv = decrease in volume, and v = original volume.

(–*ve* sign indicates that volume *decreases with increase in pressure*)

Also, volume $v \propto \frac{1}{\rho}$, or $v\,\rho = \text{constant}$

Differentiating both sides, we get

$$v\,d\rho + \rho\,dv = 0 \quad \text{or} \quad -\frac{dv}{v} = \frac{d\rho}{\rho}$$

Substituting the value of $-\frac{dv}{v}\left(=\frac{dp}{K}\right)$ from eqn. (*i*), we get:

$$\frac{dp}{K} = \frac{d\rho}{\rho} \quad \text{or} \quad \frac{dp}{d\rho} = \frac{K}{\rho}$$

Substituting this value of $\frac{dp}{d\rho}$ in eqn. (29), we get

$$\boxed{C = \sqrt{\frac{K}{\rho}}} \qquad \ldots(30)$$

Eqn. (30) *is applicable for liquids and gases.*

Sonic Velocity for Isothermal Process

For is tothermal process : $\frac{p}{\rho} = \text{constant}$

Differentiating both sides, we get:

$$\frac{\rho \,.\, dp - p \,.\, d\rho}{\rho^2} = 0 \quad \text{or} \quad \frac{dp}{\rho} - \frac{p.d\rho}{\rho^2} = 0$$

or,

$$\frac{dp}{\rho} = \frac{p \,.\, d\rho}{\rho^2} \quad \text{or} \quad \boxed{\frac{dp}{d\rho} = \frac{p}{\rho} = RT} \quad \ldots(31)$$

$$\left(\frac{p}{\rho} = RT \quad \ldots\text{equation of state}\right)$$

Substituting the value of $\frac{dp}{d\rho}$ in eqn. (29), we get:

$$\boxed{C = \sqrt{\frac{p}{\rho}} = \sqrt{RT}} \quad \ldots(32)$$

Sonic Velocity for Adiabatic Process

For isentropic (reversible adiabatic) process: $\frac{p}{\rho^{\gamma}} = \text{constant}$

or, $$p \, . \, \rho^{-\gamma} = \text{Constant}$$

Differentiating both sides, we have $p\,(-\gamma)\,.\,\rho^{-\gamma-1}\,d\rho + \rho^{-\gamma}\,dp = 0$

Dividing both sides by $\rho^{-\gamma}$, we get: $-p\,\gamma\,\rho^{-1}\,d\rho + dp = 0$ or $dp = p\,\gamma\,\rho^{-1}\,d\rho$

or, $$\frac{dp}{d\rho} = \frac{p}{\rho}\gamma = \gamma RT \qquad \left(\because \frac{p}{\rho} = RT\right)$$

Substituting the value of $\frac{dp}{d\rho}$ in eqn. (29), we get:

$$\boxed{C = \sqrt{\gamma RT}} \qquad \ldots(33)$$

The following points are worth noting :

(i) The process is assumed to be adiabatic when minor disturbances are to be propagated through air; due to ***very high*** *velocity* of disturbances/pressure waves appreciable heat transfer does not take place.

(ii) For calculation of velocity of the sound/pressure waves, **isothermal process** *is considered only when it is mentioned in the numerical problem (that the process is isothermal). When no process is mentioned in the problem, calculations are made assuming the process to be* **adiabatic.**

(iii) For an incompressible fluid, the speed of sound is infinite (*Mach number is zero*).

MACH NUMBER

The Mach number is an important parameter in dealing with the flow of compressible fluids, when elastic forces become important and predominant.

***Mach number** is defined as the square root of the ratio of the inertia force of a fluid to the elastic force.*

$$\therefore \text{ Mach number, } M = \sqrt{\frac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\frac{\rho A V^2}{KA}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C}$$

$$\left[\because \sqrt{K/\rho} = C \quad \text{...eqn. (30)}\right]$$

i.e. $$M = \frac{V}{C} \quad \text{...(34)}$$

Thus, $$M = \frac{\text{Velocity at a point in a fluid}}{\text{Velocity of sound at that point at a given instant of time}}$$

Depending on the value of Mach number, the flow can be *classified* as follows :

1. *Subsonic flow* : Mach number is *less* than 1.0 (or $M < 1$); in this case $V < C$.
2. *Sonic flow* : Mach number is equal to 1.0 (or $M = 1$); in this case $V = C$.
3. *Supersonic flow* : Mach number is *greater* than 1.0 (or $M > 1$); in this case $V > C$.

When the Mach number in flow region is slightly less to sightly greater than 1.0, the flow is termed as ***transonic flow.***

The following points are worth noting :

(i) Mach number is important in those problems in which the flow velocity is comparable with the sonic velocity (velocity of sound). It may happen in case of airplanes travelling at very high speed, projectiles, bullets etc.

(ii) If for any flow system the Mach number is less than about 0.4 the effects of compressibility may be neglected (for that flow system).

Regime	Speed				
	(Mach)	(knots)	(mph)	(km/h)	(m/s)
Re-entry speeds	>25.0	>16,537	>19,031	>30,626	>8,508
High-hypersonic	10.0–25.0	6,615–16,537	7,680–19,031	12,251–30,626	3,403–8,508
Hypersonic	5.0–10.0	3,308–6,615	3,806–7,680	6,126–12,251	1,702–3,403
Supersonic	1.3–5.0	794-3,308	915-3,806	1,470–6,126	410–1,702
Transonic	0.8–1.3	530–794	609–914	980–1,470	273–409
Subsonic	<0.8	<530	<609	<980	<273


Temperature
12 km
39,000 ft
-56°C
-67°F
Airplane Traveling at the Speed of Sound
1,056 km/h (660 mph)
Airplane Traveling at the Speed of Sound
1,216 km/h (760 mph)
Temperature
Sea Level
15°C
59°F

Example *Find the sonic velocity for the following fluids :*

(*i*) *Crude oil of specific gravity 0.8 and bulk modulus 1.5 GN/m²^2.*

(*ii*) *Mercury having a bulk modulus of 27 GN/m^2.* **(Delhi University)**

Solution. *Crude oil:* Specific gravity = 0.8

∴ Density of oil, $\rho = 0.8 \times 1000 = 800 \text{ kg/m}^3$

Bulk modulus, $K = 1.5 \ GN/\text{m}^2$

Mercury : Bulk modulus, $K = 27 \ GN/\text{m}^2$

Density of mercury, $\rho = 13.6 \times 1000 = 13600 \text{ kg/m}^3$

Sonic velocity, C_{oil}, C_{Hg} :

Sonic velocity is given by the relation :

$$C = \sqrt{\frac{K}{\rho}} \qquad \text{...[Eqn. (30)]}$$

$$\therefore \quad C_{\text{oil}} = \sqrt{\frac{1.5 \times 10^9}{800}} = \mathbf{1369.3 \text{ m/s}} \quad \textbf{(Ans.)}$$

$$C_{Hg} = \sqrt{\frac{27 \times 10^9}{13600}} = \mathbf{1409 \text{ m/s}} \quad \textbf{(Ans.)}$$

Example *An aeroplane is flying at a height of 14 km where temperature is – 45°C. The speed of the plane is corresponding to M = 2. Find the speed of the plane if R = 287 J/kg K and γ = 1.4.*

Solution. Temperature (at a height of 14 km), $t = -45°C$.

$$T = -45 + 273 = 228\ K$$

Mach number, $M = 2$

Gas constant, $R = 287$ J/kg K

$$\gamma = 1.4$$

Speed of the plane, V :

Sonic velocity, (C) is given by:

$$C = \sqrt{\gamma RT} \quad \text{(assuming the process to be } \textit{adiabatic}) \qquad \text{...[Eqn. (33)]}$$

$$= \sqrt{1.4 \times 287 \times 228} = 302.67 \text{ m/s}$$

Also, $$M = \frac{V}{C} \qquad \text{...[Eqn. (34)]}$$

or, $$2 = \frac{V}{302.67}$$

or, $$V = 2 \times 302.67 = 605.34 \text{ m/s} = \frac{605.34 \times 3600}{1000} = \mathbf{2179.2\ km/h} \text{ (Ans.)}$$

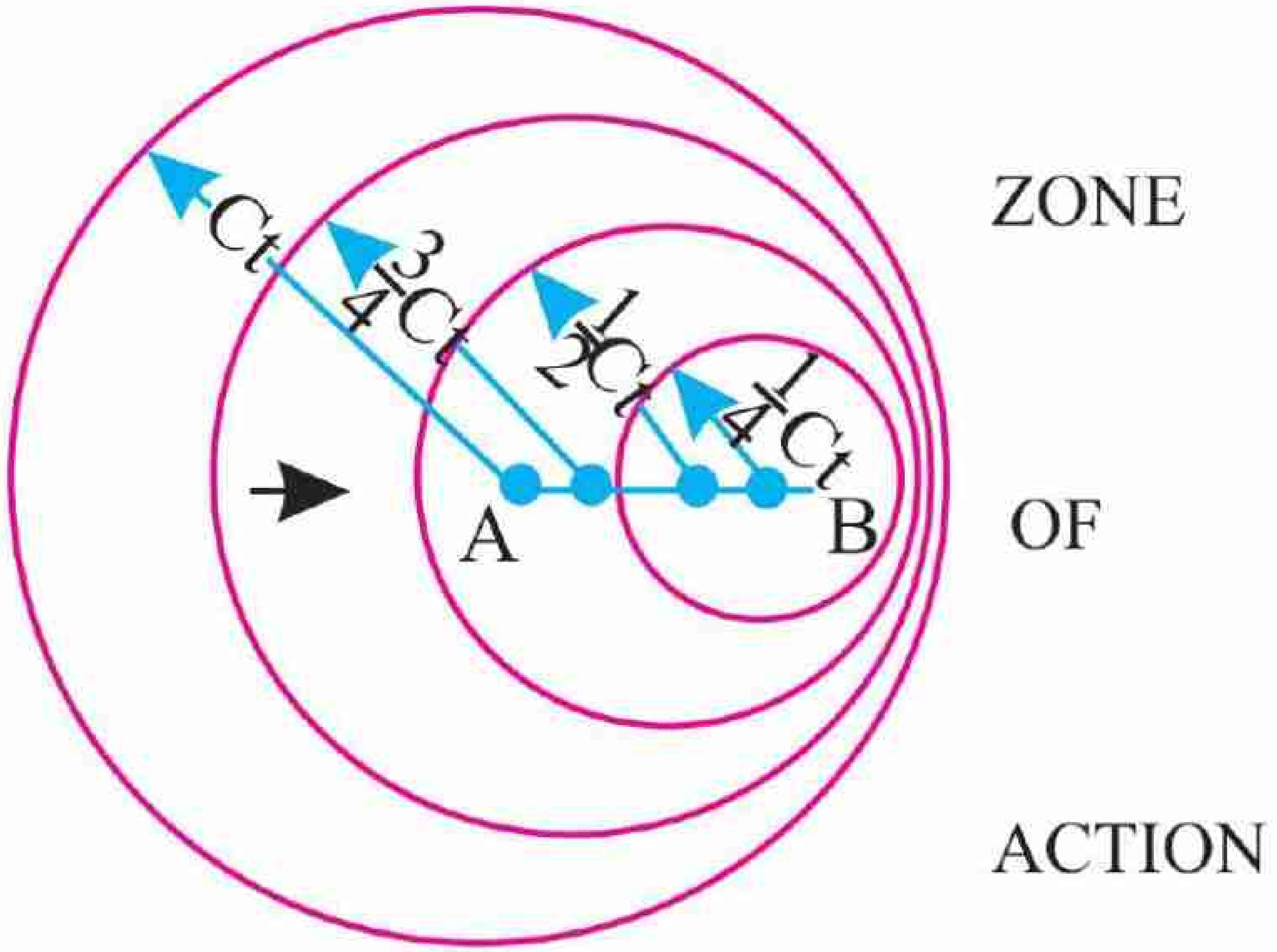
PROPAGATION OF DISTURBANCE IN COMPRESSIBLE FLUID

When some disturbance is created in a compressible fluid (elastic or pressure waves are also generated), it is propagated in all directions with sonic velocity *(= C)* and its nature of propagation depends upon the *Mach number (M)*. Such disturbance may be created when an object moves in a relatively stationary compressible fluid or when a compressible fluid flows past a stationary object.

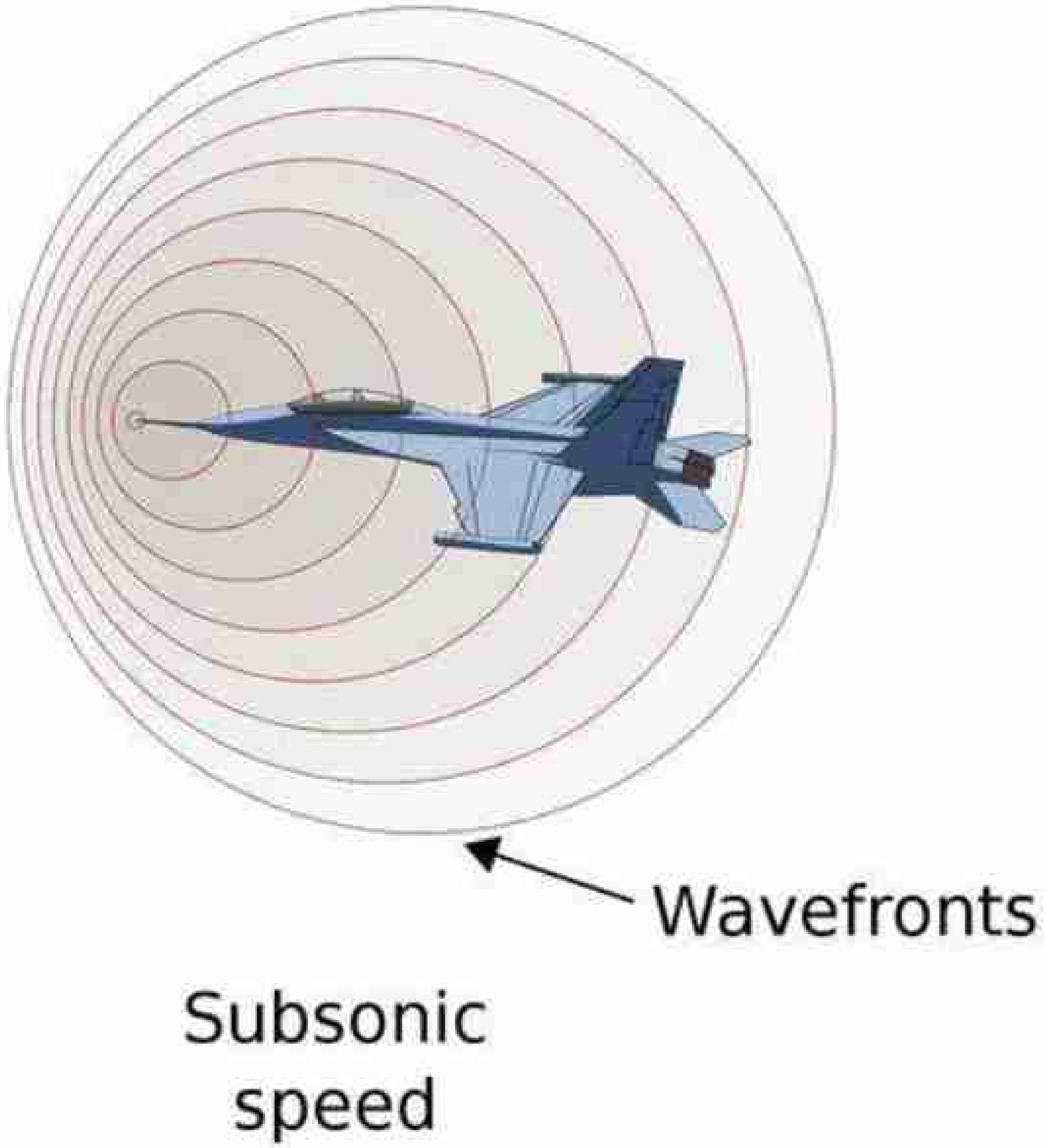
Consider a tiny projectile moving in a straight line with velocity V through a compressible fluid which is stationary. Let the projectile is at A when time $t = 0$, then in time t it will move through a distance $AB = Vt$. During this time, the disturbance which originated from the projectile when it was at A will grow into the surface of sphere of radius Ct as shown in Fig., which also shows the growth of the other disturbances which will originate from the projectile at every $t/4$ interval of time as the projectile moves from A to B.

Let us find nature of propagation of the disturbance for different Mach numbers.

Case I: When **M < 1 (*i.e.,* V < C)**. In this case since V < C the projectile lags behind the disturbance/pressure wave and hence as shown in Fig. (a) the projectile at point B lies inside the sphere of radius Ct and inside other spheres formed by the disturbances/waves started at intermediate points.

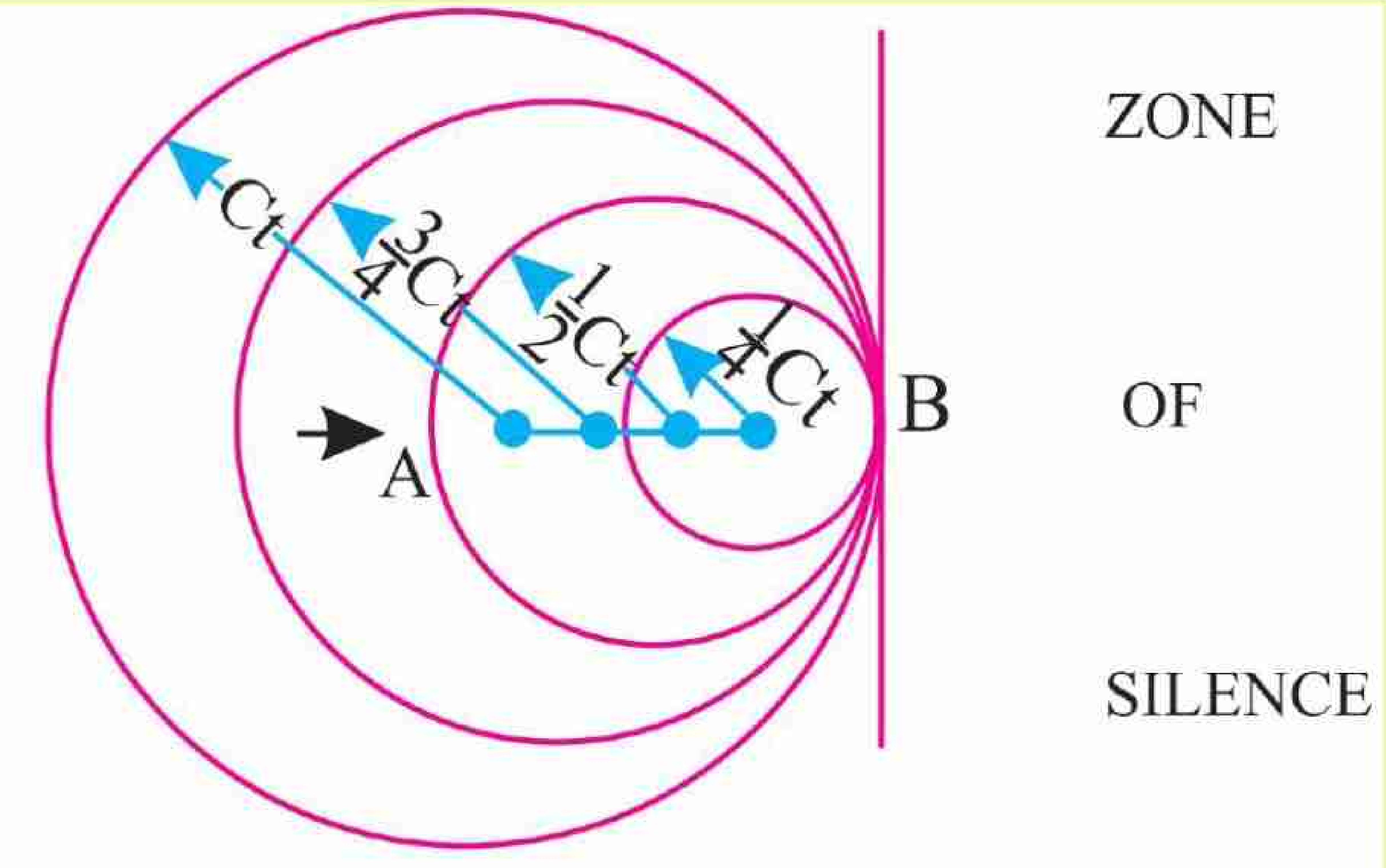


(*a*) M < 1 (V < C)
Subsonic

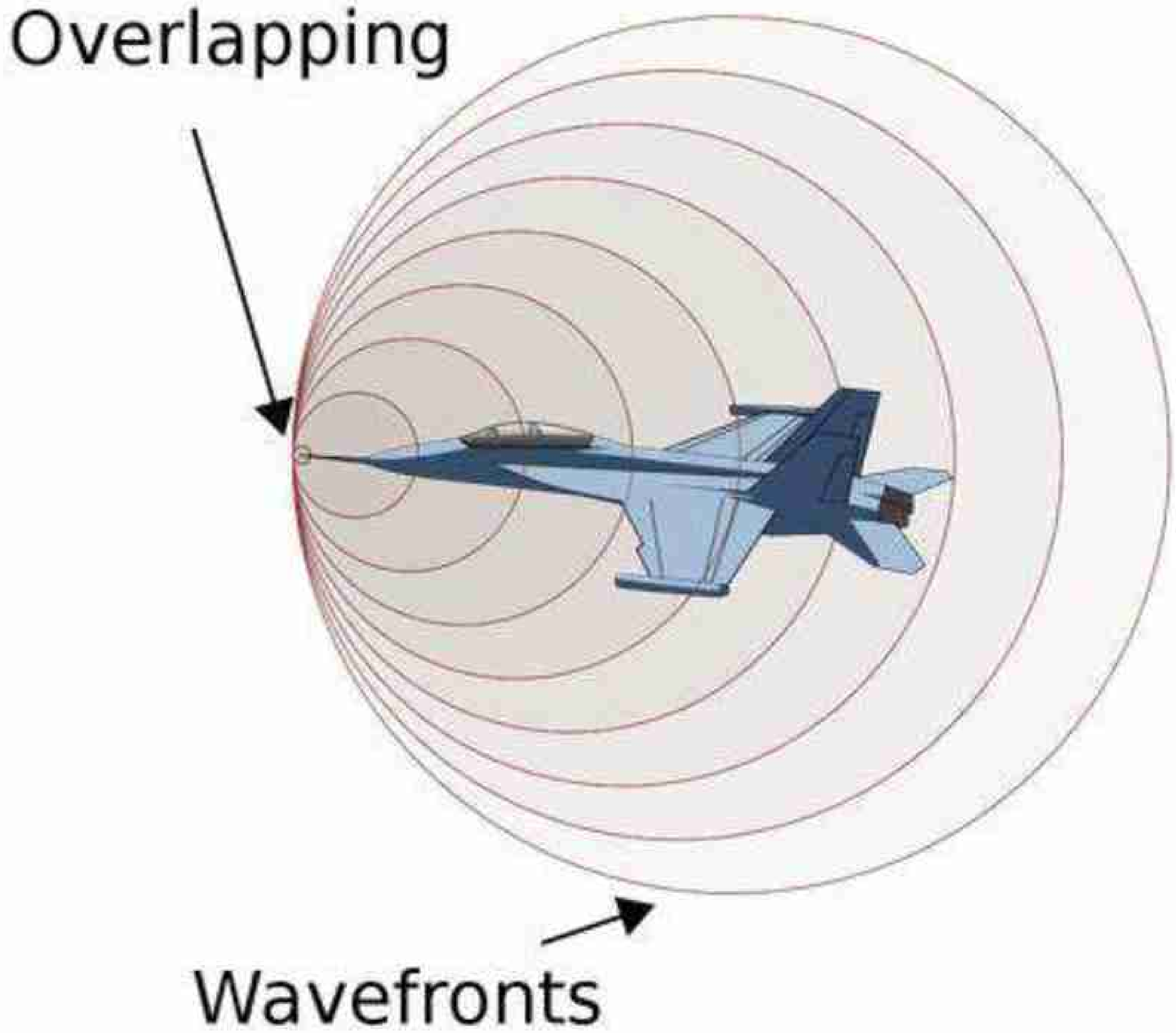


An air craft in flight creates a series of *pressure waves*, that travel outwards in all directions and are perceived as ***Sound.***

Case II: When **M = 1 (*i.e.*, V = C)**. In this case, the disturbance always travels with the projectile as shown in Fig.(b). The circle drawn with center A will pass through B.

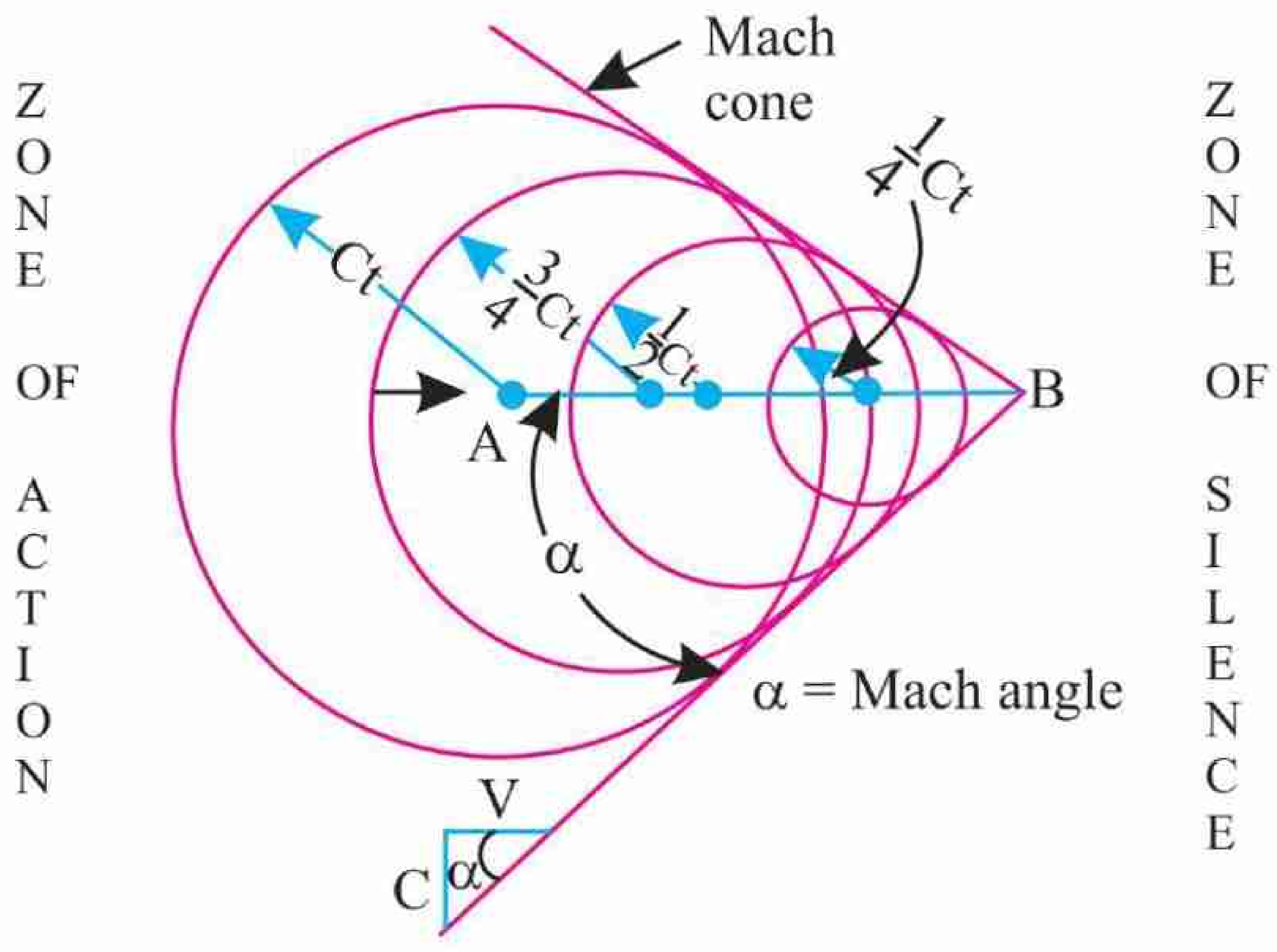


(*b*) M = 1 (V = C)
Sonic motion



The faster the air craft moves, the more compressed waves becomes

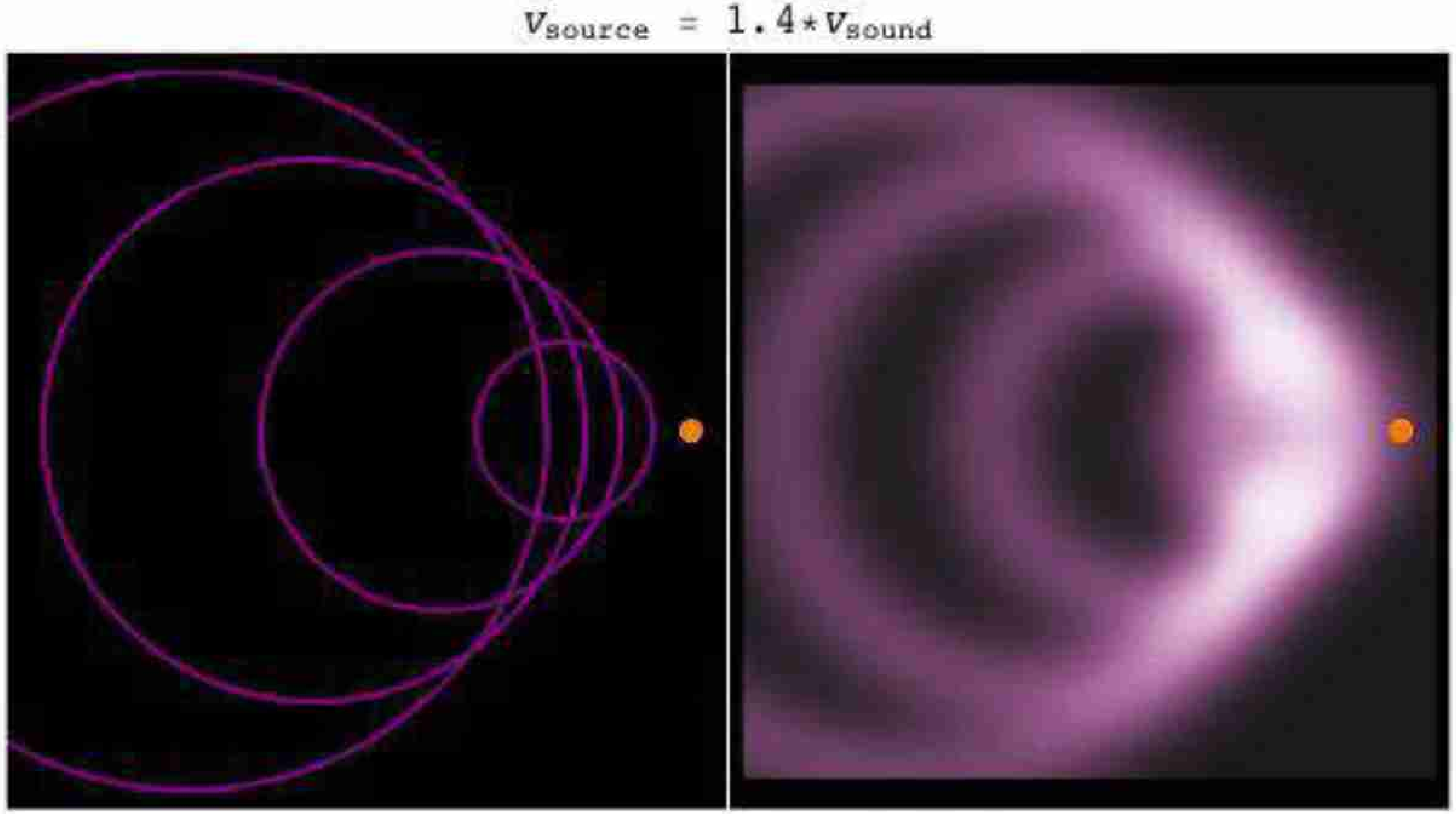
Case III: When **M > 1 (i.e., V > C)**. In this case the projectile travels faster than the disturbance. Thus, the distance AB (which the projectile has travelled) is more than *Ct*, and hence the projectile at point 'B' is outside the spheres formed due to formation and growth of disturbance at *t = 0* and at the intermediate points (Fig.(c)). If the *tangents* are drawn (from the point B) to the circles, the spherical pressure waves form a cone with its vertex at B. It is known as ***Mach cone***. The semi vertex angle α of the cone is known as ***Mach angle*** which is given by:



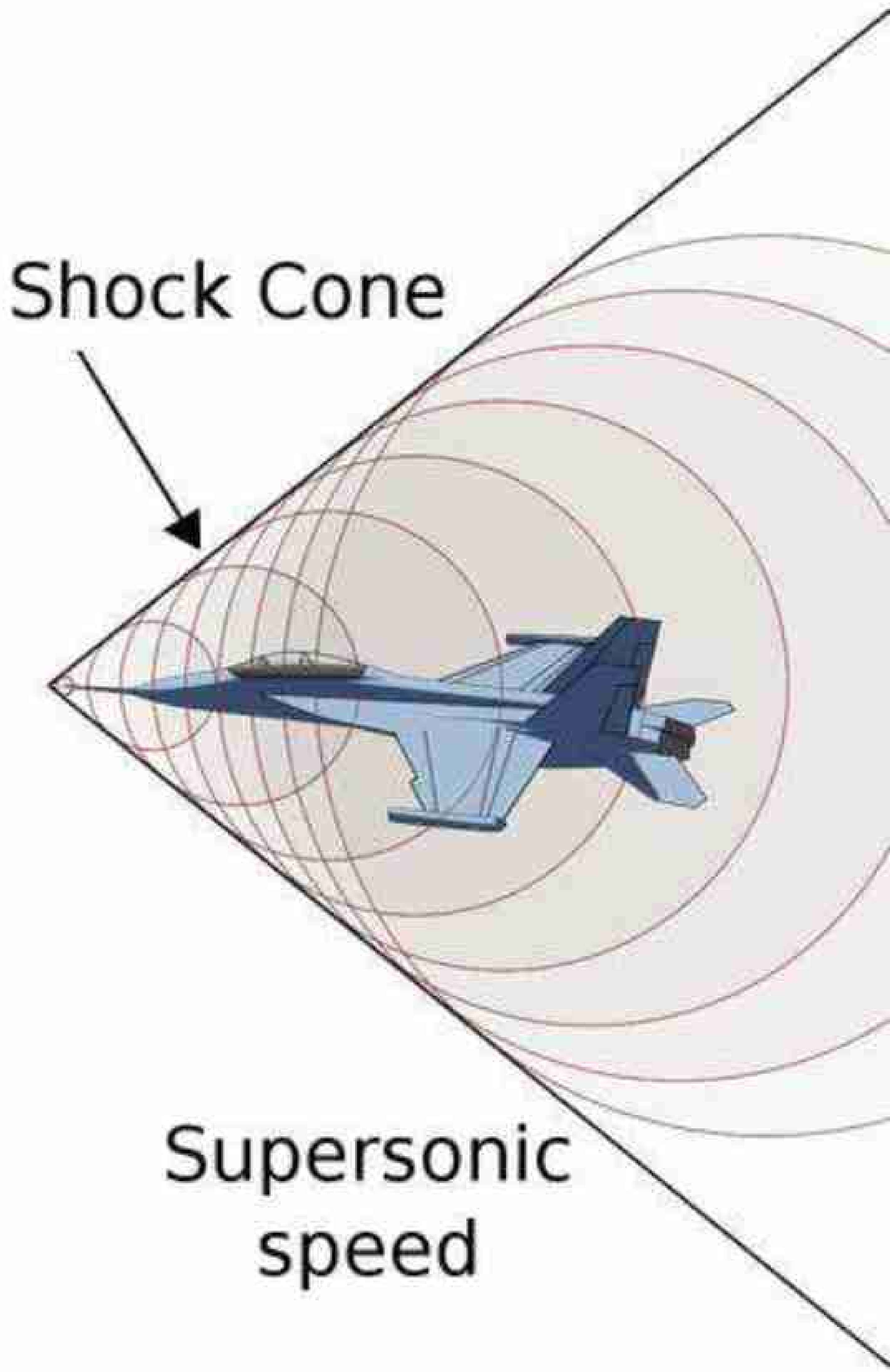
(c) M > 1 (V > C)
Supersonic motion

$$\sin\alpha = \frac{Ct}{Vt} = \frac{C}{V} = \frac{1}{M}$$

In such a case (M > 1), the effect of the disturbance is felt only in region inside the Mach cone, this region is called zone of action. The region outside the Mach cone is called zone of silence. It has been observed that *when an airplane is moving with supersonic speed, its noise is heard only after the plane has already passed over us.*



When an object is moving faster than the speed of sound, eventually the waves merge into a ***Shock wave***. A person on the ground hears a ***Boom*** when the shock wave crosses his or her location.



V =0. C

•

Vapor cone

A **vapor cone**, also known as shock collar or shock egg, is a visible cloud of *condensed water* that can sometimes form around an object moving at high speed through moist air, for example, an aircraft flying at transonic speeds. When the localized air pressure around the object drops, so does the air temperature. If the temperature drops below the saturation temperature, a cloud forms.

Example *Find the velocity of a bullet fired in standard air if its Mach angle is 40° .*

Solution. Mach angle, $\alpha = 40°$

$\gamma = 1.4$

For standard air: $R = 287$ J/kg K, $t = 15°C$ or $T = 15 + 273 = 288$ K

Velocity of the bullet, *V* :

$$\text{Sonic velocity, } C = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 288} = 340.2 \text{ m/s}$$

Now,
$$\sin \alpha = \frac{C}{V}$$

or,
$$\sin 40° = \frac{340.2}{V} \quad \text{or} \quad V = \frac{340.2}{\sin 40°} = \mathbf{529.26\ m/s}$$ **(Ans.)**

Example *A projectile is travelling in air having pressure and temperature as 88.3 kN/m^2 and – 2°C. If the Mach angle is 40°, find the velocity of the projectile.*

Take $\gamma = 1.4$ and $R = 287$ J/kg K. **[M.U.]**

Solution.

$$\text{Pressure, } p = 88.3 \text{ kN/m}^2$$

$$\text{Temperature, } T = -2 + 273 = 271 \text{ K}$$

$$\text{Mach angle, } M = 40°$$

$$\gamma = 1.4,\ R = 287 \text{ J/kg K}$$

Velocity of the projectile, *V*:

$$\text{Sonic velocity, } C = \sqrt{\gamma RT} = \sqrt{1{\cdot}4 \times 287 \times 271} \simeq 330 \text{ m/s}$$

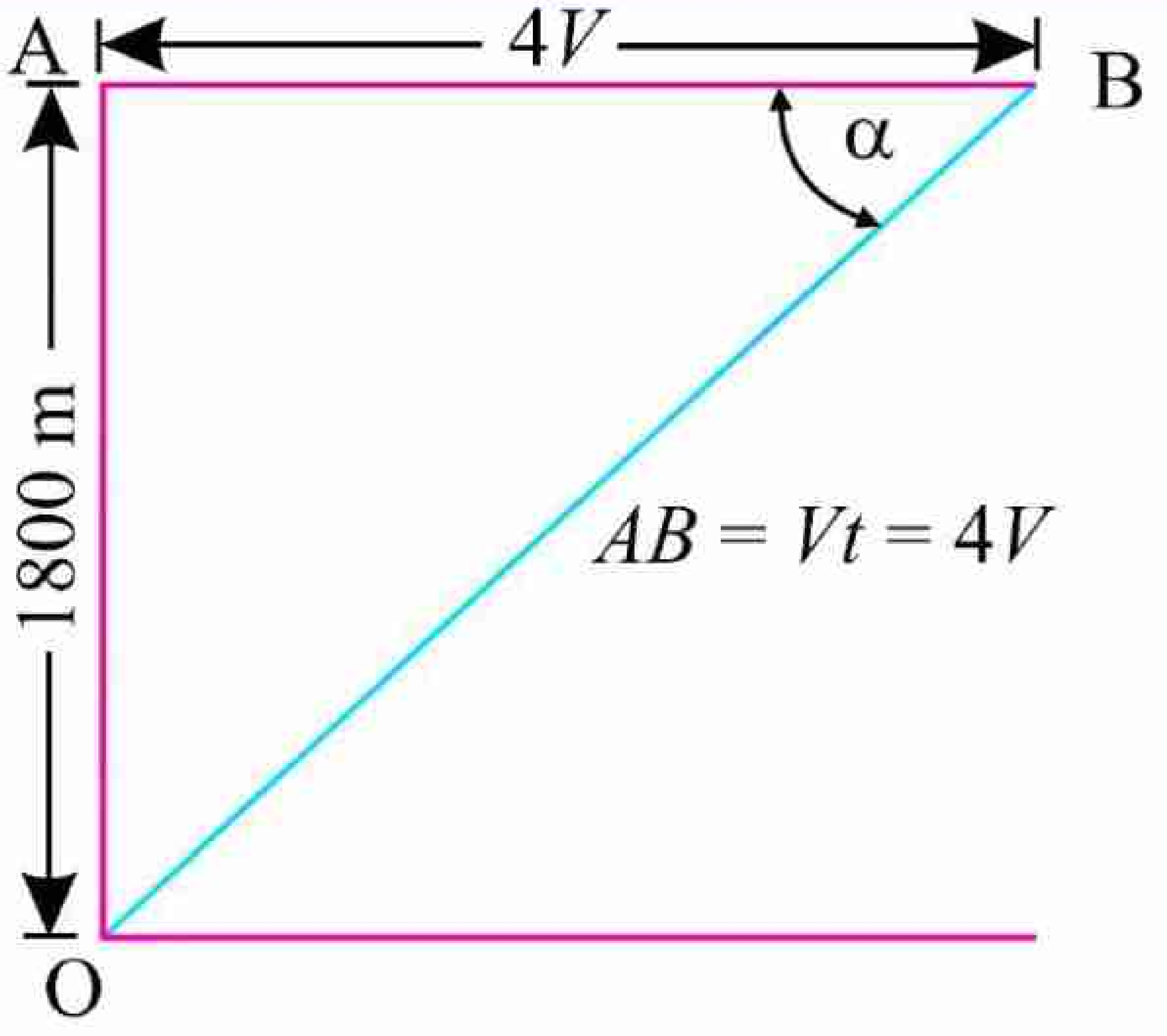
Now,

$$\sin\alpha = \frac{C}{V} \quad \text{or} \quad \sin 40° = \frac{330}{V}$$

or,

$$V = \frac{330}{\sin 40°} = \mathbf{513.4\ m/s}\ \textbf{(Ans.)}$$

Example *A supersonic aircraft flies at an altitude of 1.8 km where temperature is 4°C. Determine the speed of the aircraft if its sound is heard 4 seconds after its passage over the head of an observer. Take R = 287 J/kg K and γ = 1.4.*



Solution. Altitude of the aircraft = 1.8 km = 1800 m

$$\text{Temperature, } T = 4 + 273 = 277 \text{ K}$$

$$\text{Time, } t = 4s$$

Speed of the aircraft, V :

Refer to Fig. 15.5. Let O represent the observer and A the position of the aircraft just vertically over the observer. After 4 seconds, the aircraft reaches the position represented by the point B. Line AB represents the wave front and α the Mach angle.

From Fig. 15.5, we have :

$$\tan\alpha = \frac{1800}{4V} = \frac{450}{V} \quad \text{...(i)}$$

But, Mach number $M = \dfrac{V}{C} = \dfrac{1}{\sin\alpha}$

or,

$$V = \frac{C}{\sin\alpha} \quad \text{...(ii)}$$

Substituting the value of V in eqn. (i), we get:

$$\tan\alpha = \frac{450}{(C/\sin\alpha)} = \frac{450\sin\alpha}{C}$$

or,
$$\frac{\sin\alpha}{\cos\alpha} = \frac{450\sin\alpha}{C} \quad \text{or} \quad \cos\alpha = \frac{C}{450} \qquad ...(iii)$$

But,
$$C = \sqrt{\gamma RT}\text{ , where } C \text{ is the sonic velocity}$$

$$R = 287 \text{ J/kg K and } \gamma = 1.4 \qquad ...(Given)$$

$$\therefore \quad C = \sqrt{1.4 \times 287 \times 277} = 333.6 \text{ m/s}$$

Substituting the value of C in eqn. (ii), we get:

$$\cos\alpha = \frac{333.6}{450} = 0.7413$$

$$\therefore \quad \sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - 0.7413^2} = 0.6712$$

Substituting the value of $\sin\alpha$ in eqn. (ii), we get:

STAGNATION PROPERTIES

The point on the immersed body where the velocity is zero is called stagnation point. At this point velocity head is converted into pressure head. The values of pressure (p_s), temperature (T_s) and density (ρ_s) at stagnation point are called ***stagnation properties***.

During a stagnation process, the kinetic energy of a fluid is converted to enthalpy (internal energy + flow energy), which results in an increase in the fluid temperature and pressure.

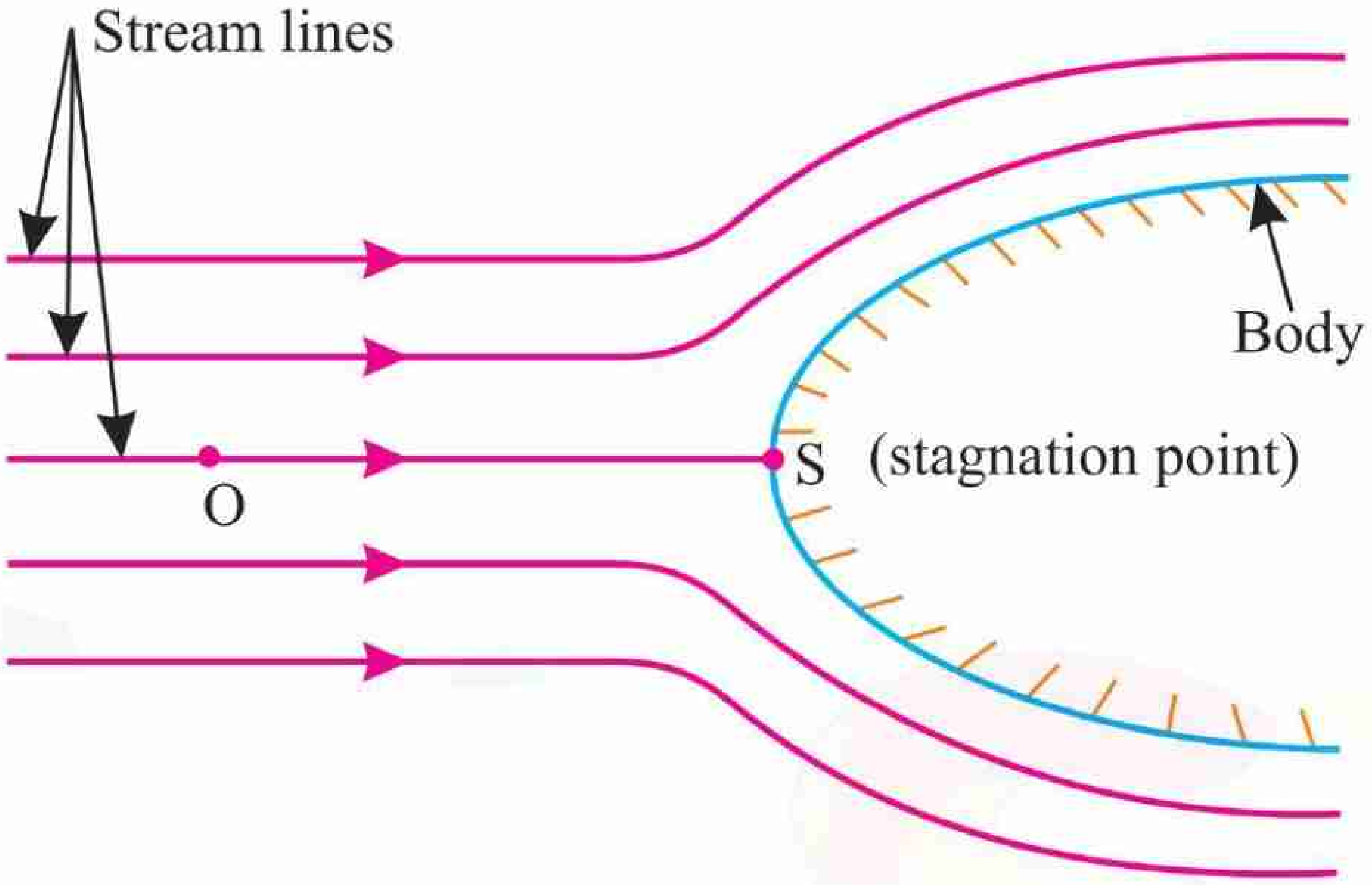


Fig. Stagnation properties.

@V.Manikanth, Asst. Prof., Dept. of Mechanical, SRKREC(A).

Expression for Stagnation Pressure (p_s) in Compressible Flow

Consider the flow of compressible fluid past an immersed body where the velocity becomes zero. Consider *frictionless adiabatic (isentropic)* condition. Let us consider two points, O in the free stream and the stagnation point S.

Let, p_0 = Pressure of compressible fluid at point ***O***,

V_0 = Velocity of fluid at O,

ρ_0 = Density of fluid at O,

T_0 = Temperature of fluid at O,

and p_s,**Vs**, ρ_s and T_s corresponding values of pressure, velocity density, and temperature at point ***S***.

Applying Bernoulli's equation for adiabatic (frictionless) flow at points O and S, (given by eqn.26), we get:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0 g}+\frac{V_0^2}{2g}+z_0 = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_s}{\rho_s g}+\frac{V_s^2}{2g}+z_s$$

But $z_0 = z_s$; the above equation reduces to:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0 g}+\frac{V_0^2}{2g} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_s}{\rho_s g}+\frac{V_s^2}{2g}$$

Cancelling 'g' on both the sides, we have:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0}+\frac{V_0^2}{2} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_s}{\rho_s}+\frac{V_s^2}{2}$$

At point S the velocity is zero, i.e. $V_s = 0$; the above equation becomes:

$$\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_0}{\rho_0}-\frac{p_s}{\rho_s}\right) = -\frac{V_0^2}{2}$$

or, $$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0}\left(1-\frac{p_s}{\rho_s}\times\frac{\rho_0}{p_0}\right) = -\frac{V_0^2}{2}$$

or, $$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0}\left(1-\frac{p_s}{p_0}\times\frac{\rho_0}{\rho_s}\right) = -\frac{V_0^2}{2} \quad ...(i)$$

For adiabatic process:

$$\frac{p_0}{\rho_0^{\gamma}} = \frac{p_s}{\rho_s^{\gamma}} \quad \text{or} \quad \frac{p_0}{p_s} = \frac{\rho_0^{\gamma}}{\rho_s^{\gamma}}$$

or,
$$\frac{\rho_0}{\rho_s} = \left(\frac{p_0}{p_s}\right)^{\frac{1}{\gamma}} \qquad ...(ii)$$

Substituting the value of $\frac{\rho_0}{\rho_s}$ in eqn. (i), we get:

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0}\left[1-\frac{p_s}{p_0}\times\left(\frac{p_0}{p_s}\right)^{\frac{1}{\gamma}}\right] = -\frac{V_0^2}{2}$$

or,
$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0}\left[1-\left(\frac{p_s}{p_0}\right)^{1-\frac{1}{\gamma}}\right] = -\frac{V_0^2}{2}$$

or,
$$\left[1-\left(\frac{p_s}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right] = -\frac{V_0^2}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho_0}{p_0}$$

or, $$1+\frac{V_0^2}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho_0}{p_0} = \left(\frac{p_s}{p_0}\right)^{\frac{\gamma-1}{\gamma}} \quad \text{...(iii)}$$

For adiabatic process, the sonic velocity is given by:

$$C = \sqrt{\gamma RT} = \sqrt{\gamma\frac{p}{\rho}} \qquad \left(\because \frac{p}{\rho} = RT\right)$$

For point 0, $$C_0 = \sqrt{\gamma\frac{p_0}{\rho_0}} \quad \text{or} \quad C_0^2 = \gamma\frac{p_0}{\rho_0}$$

Substituting the value of $\frac{\gamma p_0}{\rho_0} = C_0^2$ in eqn. (*iii*), we get:

$$1+\frac{V_0^2}{2}(\gamma-1)\times\frac{1}{C_0^2} = \left(\frac{p_s}{p_0}\right)^{\frac{\gamma-1}{\gamma}}$$

or, $$1+\frac{V_0^2}{2C_0^2}(\gamma-1) = \left(\frac{p_s}{p_0}\right)^{\frac{\gamma-1}{\gamma}}$$

$$1 + \frac{M_0^2}{2}(\gamma - 1) = \left(\frac{p_s}{p_0}\right)^{\frac{\gamma-1}{\gamma}} \qquad \left(\because \frac{V_0^2}{C_0^2} = M_0^2\right)$$

or, $$\left(\frac{p_s}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \left[1 + \left(\frac{\gamma-1}{2}\right) M_0^2\right]$$

or, $$\frac{p_s}{p_0} = \left[1 + \left(\frac{\gamma-1}{2}\right) M_0^2\right]^{\frac{\gamma}{\gamma-1}} \qquad ...(iv)$$

or, $$\boldsymbol{p_s = p_0 \left[1 + \left(\frac{\gamma-1}{2}\right) M_0^2\right]^{\frac{\gamma}{\gamma-1}}} \qquad ...(36)$$

Eqn. (36) gives the value of ***stagnation pressure.***

Compressibility correction factor:

If the right hand side of Eqn. (36) is expanded by the binomial theorem, we get:

$$p_s = p_0\left[1 + \frac{\gamma}{2}M_0^2 + \frac{\gamma}{8}M_0^4 + \frac{\gamma(2-\gamma)}{48}M_0^6\right]$$

$$= p_0\left[1 + \frac{\gamma M_0^2}{2}\left(1 + \frac{M^2}{4} + \frac{2-\gamma}{24}M_0^4 + \ldots\right)\right]$$

or,

$$p_s = p_0 + \frac{p_0\gamma M_0^2}{2}\left(1 + \frac{M_0^2}{4} + \frac{2-\gamma}{24}M_0^4 + \ldots\right) \qquad \ldots(37)$$

But,

$$M_0^2 = \frac{V_0^2}{C_0^2} = \frac{V_0^2}{\left(\frac{\gamma p_0}{\rho_0}\right)} = \frac{V_0^2\rho_0}{\gamma p_0} \qquad \left(\because C_0^2 = \frac{\gamma p_0}{\rho_0}\right)$$

Substituting the value of M_0^2 in eqn. 37, we get:

$$p_s = p_0 + \frac{p_0 \gamma}{2} \times \frac{V_0^2 \rho_0}{\gamma p_0} \left(1 + \frac{M_0^2}{4} + \frac{2-\gamma}{24} M_0^4 + \ldots \right)$$

or,

$$p_s = p_0 + \frac{\rho_0 V_0^2}{2} \left(1 + \frac{M_0^2}{4} + \frac{2-\gamma}{24} M_0^4 + \ldots \right) \qquad \ldots(38)$$

Also,

$$\boldsymbol{p_s} = p_0 + \frac{\rho_0 V_0^2}{2} \; (\textit{when compressibility effects are neglected}) \qquad \ldots(39)$$

Limits of incompressibility

The comparison of eqns. (38) and (39) shows that the effects of compressibility are isolated in the bracketed quantity and that these effects *depend only* upon the *Mach number*. The bracketed quantity $\left[i.e., \left(1 + \frac{M_0^2}{4} + \frac{2-\gamma}{24} M_0^4 + \ldots \right) \right]$ may thus be considered as a **compressibility correction factor**. It is worth noting that :

- For $M < 0.2$, the compressibility affects the pressure difference $(p_s - p_0)$ by *less than 1 per cent* and the simple formula for flow at constant density is then sufficiently accurate.
- *For larger value of M*, as the terms of binomial expansion become significant, the *compressibility effect must be taken into account.*
- When the Mach number exceeds a value of about 0.3 the *Pitot-static tube used for measuring aircraft speed needs calibration to take into account the compressibility effects.*

Expression for Stagnation Density (ρ_s):

From eqn. (*ii*), we have:

$$\frac{\rho_0}{\rho_s} = \left(\frac{p_0}{p_s}\right)^{\frac{1}{\gamma}} \quad \text{or} \quad \frac{\rho_s}{\rho_0} = \left(\frac{p_s}{p_0}\right)^{\frac{1}{\gamma}} \quad \text{or} \quad \rho_s = \rho_0 \left(\frac{p_s}{p_0}\right)^{\frac{1}{\gamma}}$$

Substituting the value of $\left(\frac{p_s}{p_0}\right)$ from eqn. (*iv*), we get:

$$p_s = \rho_0 \left[\left\{1 + \left(\frac{\gamma - 1}{2}\right) M_0^2\right\}^{\frac{\gamma}{\gamma - 1}}\right]^{\frac{1}{\gamma}}$$

$$\boldsymbol{\rho_s = \rho_0 \left[1 + \left(\frac{\gamma - 1}{2}\right) M_0^2\right]^{\frac{1}{\gamma - 1}}} \qquad \text{...(40)}$$

Expression for stagnation temperature (T_s)

The equation of state is given by : $\frac{p}{\rho} = RT$

For stagnation point, the equation of state may be written as :

$$\frac{p_s}{\rho_s} = RT_s \quad \text{or} \quad T_s = \frac{1}{R}\frac{p_s}{\rho_s}$$

Substituting the values of p_s and ρ_s from eqns. (36) and (40) we get:

$$T_s = \frac{1}{R}\frac{p_0\left[1+\left(\frac{\gamma-1}{2}\right)M_0^2\right]^{\frac{\gamma}{\gamma-1}}}{p_0\left[1+\left(\frac{\gamma-1}{2}\right)M_0^2\right]^{\frac{1}{\gamma-1}}}$$

$$= \frac{1}{R}\frac{p_0}{\rho_0}\left[1+\left(\frac{\gamma-1}{2}\right)M_0^2\right]^{\left(\frac{\gamma}{\gamma-1}-\frac{1}{\gamma-1}\right)}$$

$$= \frac{1}{R}\frac{p_0}{\rho_0}\left[1+\left(\frac{\gamma-1}{2}\right)M_0^2\right]^{\left(\frac{\gamma-1}{\gamma-1}\right)}$$

or,

$$\boldsymbol{T_s = T_0\left[1+\left(\frac{\gamma-1}{2}\right)M_0^2\right]} \qquad \text{...(41)}$$

$$\left(\because \frac{p_0}{\rho_0} = RT_0\right)$$

Example *An aeroplane is flying at 1000 km/h through still air having a pressure of 78.5 kN/m² (abs.) and temperature – 8° C . Calculate on the stagnation point on the nose of the plane :*

(*i*) *Stagnation pressure,*

(*ii*) *Stagnation temperature, and*

(*iii*) *Stagnation density.*

Take for air : R = 287 J/kg K and γ = 1.4.

Solution. Speed of aeroplane,

$$V = 1000 \text{ km/h} = \frac{1000 \times 1000}{60 \times 60} = 277{\cdot}77 \text{ m/s}$$

$$\text{Pressure of air, } p_0 = 78.5 \text{ kN/m}^2$$

$$\text{Temperature of air, } T_0 = -8 + 273 = 265 \text{ K}$$

For air : $R = 287 \text{ J/kg K}, \gamma = 1.4$

The sonic velocity for adiabatic flow is given by:

$$C_0 = \sqrt{\gamma R T_0} = \sqrt{1.4 \times 287 \times 265} = 326.31 \text{ m/s}$$

$$\therefore \quad \text{Mach number, } M_0 = \frac{V_0}{C_0} = \frac{277.77}{326.31} = 0.851$$

(*i*) Stagnation pressure, p_s :

The stagnation pressure (p_s) is given by the relation:

$$p_s = p_0 \left[1 + \left(\frac{\gamma - 1}{2}\right) M_0^2\right]^{\frac{\gamma}{\gamma - 1}} \quad \text{...[Eqn. (36)]}$$

or,

$$p_s = 78.5 \left[1 + \left(\frac{1.4 - 1}{2}\right) \times 0.851^2\right]^{\frac{1.4}{1.4 - 1}}$$

$$= 78.5\,(1.145)^{3.5} = \mathbf{126.1\ kN/m^2} \quad \textbf{(Ans.)}$$

(*ii*) Stagnation temperature, T_s :

The stagnation temperature is given by:

$$T_s = T_0 \left[1 + \left(\frac{\gamma - 1}{2}\right) M_0^2\right] \quad \text{...[Eqn. (41)]}$$

$$= 265 \left[1 + \frac{1.4 - 1}{2} \times 0.851^2\right] = 303.4 \text{ K} \quad \text{or} \quad \mathbf{30.4° C} \quad \textbf{(Ans.)}$$

(iii) Stagnation density, ρ_s :

The stagnation density (ρ_s) is given by:

$$\frac{p_s}{\rho_s} = RT_s \quad \text{or} \quad \rho_s = \frac{p_s}{RT_s}$$

or,

$$p_s = \frac{126.1 \times 10^3}{287 \times 303.4} = \mathbf{1.448\ kg/m^3}\ \textbf{(Ans.)}$$

Example *Air at a pressure of 220 kN/m^2 and temperature 27°C is moving at a velocity of 200 m/s. Calculate the stagnation pressure if*

(i) *Compressibility is neglected;* **(ii)** *Compressibility is accounted for.*

For air, take $R = 287$ J/kg K, $\gamma = 1.4$

Solution. Pressure of air, p_0 = 200 kN/m^2

Temperature of air, T_0 = 27 + 233 = 300 K

Velocity of air, V_0 = 200 m/s

Stagnation pressure, p_s :

(*i*) Compressibility is neglected :

$$p_s = p_0 + \frac{\rho_0 V_0^2}{2}$$

where, $$\rho_0 = \frac{p_0}{RT_0} = \frac{220 \times 10^3}{287 \times 300} = 2.555 \text{ kg/m}^3$$

$\therefore$ $$p_s = 220 + \frac{2.555 \times 200^2}{2} \times 10^{-3} \text{ (kN/m}^2) = \mathbf{271.1 kN / m^2} \quad \textbf{(Ans.)}$$

(*ii*) Compressibility is accounted for :

The stagnation pressure, when compressibility is accounted for, is given by:

$$p_s = p_0 + \frac{\rho_0 V_0^2}{2}\left(1 + \frac{M_0^2}{4} + \frac{2-\gamma}{24} M_0^4 + ...\right) \quad \text{...[Eqn. (38)]}$$

$$\text{Mach number, } M_0 = \frac{V_0}{C_0} = \frac{200}{\sqrt{\gamma R T_0}} = \frac{200}{\sqrt{1.4 \times 287 \times 300}} = 0.576$$

Whence, $$p_s = 220 + \frac{2.555 \times 200^2}{2} \times 10^{-3}\left(1 + \frac{0.576^2}{4} + \frac{2-1.4}{24} \times 0.576^4\right)$$

or, $$p_s = 220 + 51.1\,(1 + 0.0829 + 0.00275) = \mathbf{275.47 \text{ kN/m}^2} \textbf{ (Ans.)}$$

AREA-VELOCITY RELATIONSHIP AND EFFECT OF VARIATION OF AREA FOR SUBSONIC, SONIC AND SUPERSONIC FLOWS:

For an *incompressible flow* the continuity equation may be expressed as :

$$AV = \text{Constant, which when differentiated gives,}$$

$$AdV + VdA = 0$$

or,

$$\frac{dA}{A} = -\frac{dV}{V} \quad ...(42)$$

But in case of *compressible flow*, the continuity equation is given by:

$$\rho AV = \text{Constant, which can be differentiated to give}$$

$$\rho d(AV) + AVd\rho = 0$$

or, $$\rho(AdV + VdA) + AVd\rho = 0$$

or, $$\rho AdV + \rho VdA + AVd\rho = 0$$

Dividing both sides by ρAV, we get:

$$\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0 \quad ...(43)$$

or, $$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho} \quad \text{...(43 (a)}$$

The Euler's equation for compressible fluid is given by:

$$\frac{dp}{\rho} + VdV + gdz = 0$$

Neglecting the z terms the above equation reduces to:

$$\frac{dp}{\rho} + VdV = 0$$

This equation can also be expressed as:

$$\frac{dp}{\rho} \times \frac{d\rho}{d\rho} + VdV = 0$$

or, $$\frac{dp}{d\rho} \times \frac{d\rho}{\rho} + VdV = 0$$

But, $$\boxed{\frac{dp}{d\rho} = C^2} \quad \text{...[Eqn. (29)]}$$

$$\therefore \qquad C^2 \times \frac{d\rho}{\rho} + VdV = 0$$

$$\text{or,} \qquad C^2 \frac{d\rho}{\rho} = -VdV \quad \text{or} \quad \frac{d\rho}{\rho} = -\frac{VdV}{C^2}$$

Substituting the value of $\frac{d\rho}{\rho}$ in eqn. (43), we get:

$$\frac{dV}{V} + \frac{dA}{A} - \frac{VdV}{C^2} = 0$$

$$\text{or,} \qquad \frac{dA}{A} = \frac{VdV}{C^2} - \frac{dV}{V} = \frac{dV}{V}\left(\frac{V^2}{C^2} - 1\right)$$

$$\therefore \qquad \boxed{\frac{dA}{A} = \frac{dV}{V}\left(M^2 - 1\right)} \qquad \left(\because M = \frac{V}{C}\right) \qquad \text{...(44)}$$

This important equation is due to ***Hugoniot.***

Eqns. (42) and (44) give variation of $\left(\frac{dA}{A}\right)$ for the flow of incompressible and compressible fluids respectively. The ratios $\left(\frac{dA}{A}\right)$ and $\left(\frac{dV}{V}\right)$ are respectively fractional variations in the values of area and flow velocity in the flow passage.

Further, in order to study the variation of pressure with the change in flow area, an expression similar to eqn. (44), as given below, can be obtained:

$$dp = \rho V^2 \left(\frac{1}{1 - M^2}\right) \frac{dA}{A} \quad ...(45)$$

From eqns. (44) and (45), it is possible to formulate the following *conclusions* of practical significance:

(*i*) For subsonic flow ($M < 1$) :

$$\frac{dV}{V} > 0 \; ; \frac{dA}{A} < 0 \; ; dp < 0 \text{ (convergent nozzle)}$$

$$\frac{dV}{V} < 0 \; ; \frac{dA}{A} > 0 \; ; dp > 0 \text{ (divergent diffuser)}$$

(*ii*) For supersonic flow ($M > 1$) :

$$\frac{dV}{V} > 0 \; ; \frac{dA}{A} > 0 \; ; dp < 0 \text{ (divergent nozzle)}$$

$$\frac{dV}{V} < 0 \; ; \frac{dA}{A} < 0 \; ; dp > 0 \text{ (convergent diffuser)}$$

(*iii*) For sonic flow ($M = 1$) :

$\frac{dA}{A}$ = 0 (straight flow passage since dA must be zero)

and, dp = (zero/zero) *i.e.* indeterminate, but when evaluated, the change of pressure $dp = 0$, since $dA = 0$ and the flow is frictionless.

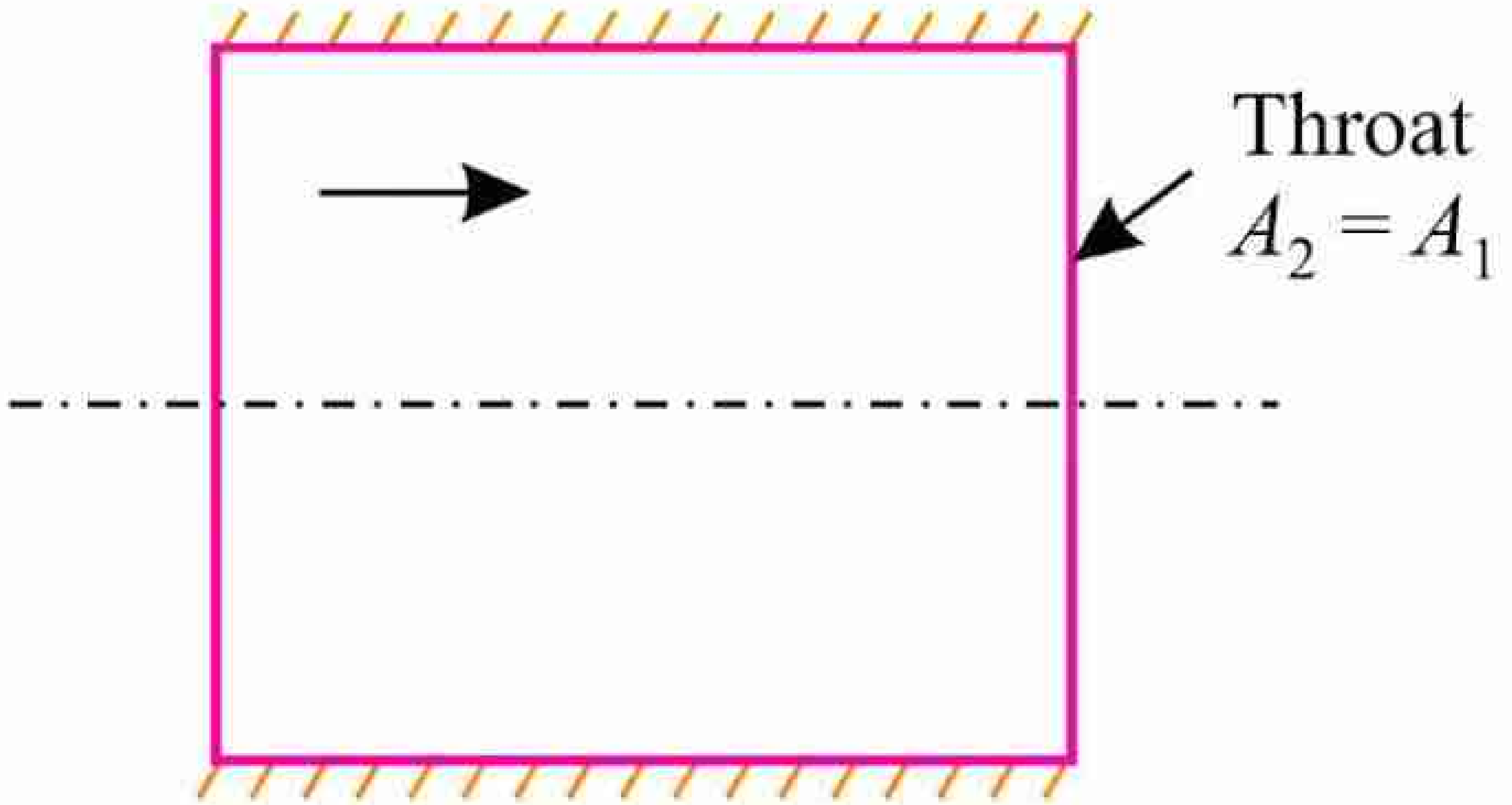
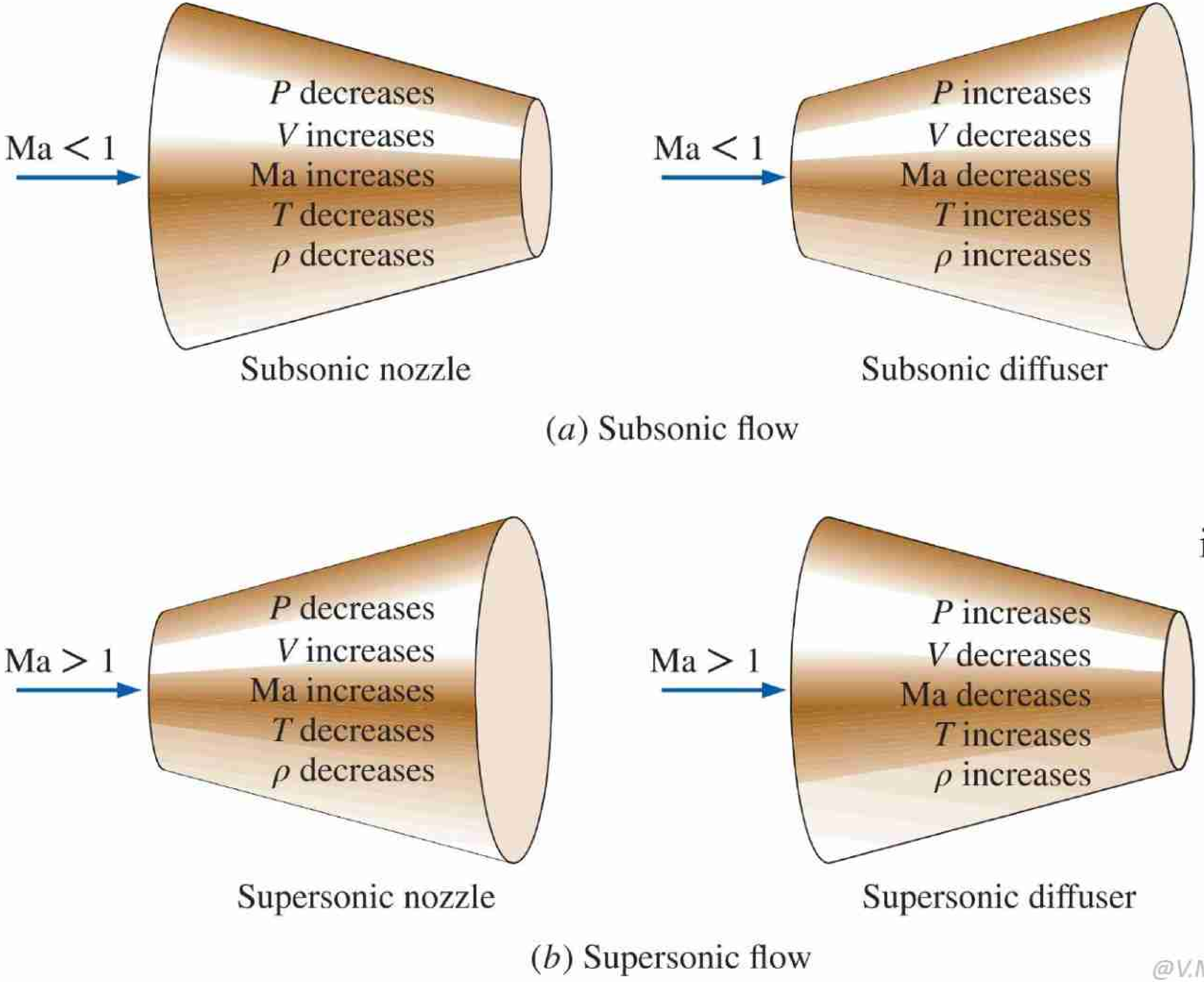


Fig. Sonic Flow (M = 1)



FIGURE

Variation of flow properties in subsonic and supersonic nozzles and diffusers.